Towards a New Order of the Polyhedral Honeycombs: Part II: Who Dances with Whom?

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Abstract

This paper investigates the ordering of the polyhedra that comprise the periodic polyhedral honeycombs, by considering how pairs of polyhedra regularly combine or mate, proximally or distally, along $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{3}$ axes of reference cubic and tetrahedral lattices - first for pairs of Great Enablers (GEs), the positive and negative tetrahedra and truncated tetrahedra; then for pairs of GEs and the eight Primary Polytopes (PPs); and then for pairs of PPs. The three types of mating, $GE{:}GE$, $GE{:}PP$ and $PP{:}PP$, correlate with the three symmetry groups {2,3,3|2,3,3}, {2,3,3|2,3,4} and {2,3,4|2,3,4}, respectively, of the arrays; and these matings typically occur in pairs, which display a one-to-one correspondence with the possible periodic honeycombs. I formally differentiate the PPs into two groups of four, which lays the groundwork for a proposed new order of the honeycombs.

Keywords: polyhedra, honeycomb, tessellation, spatial harmony, form, order.

1 Introduction: Ordering the Primary Polytopes

Part I of this series [1], following on from my earlier research with the polyhedra [2, 3], identified four Great Enablers i.e. $GE{:}\{T^{+/-}, D^{+/-}\} = \{T^+, T^-, D^+, D^-\}$; eight Primary Polytopes, including the 0-D vertex VT i.e. $PP{:}\{VT, TO, SR, GR, OH, CO, CB, TC\}$; and three Neutral Elements, considered in a restricted sense, where they develop as 3-D polyhedra, as $NE_{rst} = \{SP, RP, OP\}$; or in a complete sense, as secondary Neutral Elements of 0-D, 1-D, 2-D or 3-D polytopes $NE_{exp}{:}\{vt, ae, sq, og, SP, RP, OP\}$. That provided an overview of the honeycombs they form. In this paper, I seek to discern a natural order among the various polytopes, according to how they relate to one another by their ability to meet along relevant axes; and in relation to the periodic structure of the honeycombs. How do GEs relate one to another; how do they relate to PPs; and how do PPs relate one to another *sans* the GEs?

Section 2 discusses axial mating of polytopes along $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{3}$ axes of reference cubic and tetrahedral lattices. Section 3 investigates $GE{:}GE$ mating, yielding the {2,3,3|2,3,3} honeycomb. Section 4 explores $GE{:}PP$ mating and the four {2,3,3|2,3,4} honeycombs. Section 5 considers $PP{:}PP$ mating, the ten {2,3,4|2,3,4} honeycombs, and formally differentiates two groups of four PPs. I then propose further research on a formal model.

2 Axial Relationships of Pairs of Polytopes

As with the reference cubic and tetrahedral lattices of the honeycombs, individual PP are situated within an orthogonal reference system. Their relationships along the XYZ axes, diagonal axes, and long diagonal axes of the cube and cubic lattice - along their $\sqrt{1}, \sqrt{2}, \sqrt{3}$ axes - can be determined on the basis of whether or not they are compatible: can they mate together, at either a vertex, a transverse (or possibly axial) edge, or an axial (or possibly transverse) face? This mating may be *proximal*, where they make actual contact (vertex, edge or face); or *distal*, i.e. through a secondary neutral intermediary element (which might be an axial edge, neutral face, or neutral polygon (a prism)). CB and VT do not mate (square-to-vertex) along a $\sqrt{1}$ axis, nor (edge-to-vertex) along a $\sqrt{2}$ axis; but do (vertex-to-vertex) along a $\sqrt{3}$ axis. TO and OH do not mate (rotated square-to-vertex) along a $\sqrt{1}$ axis, nor (hexagon-to-triangle) along a $\sqrt{3}$ axis; but do so (diagonal transverse edge-to-edge) along a $\sqrt{2}$ axis.

3 Mating of GE:GE pairs with one another

First, the GEs differ from the PPs and NEs, in that they only develop {2,3,3} symmetry on the $\sqrt{1}$, and on alternating α and β tetrahedral $\sqrt{3}$ axes. *Please note that all the figures and tables of this paper are shown at http://rmeurant.com/its/hn2.html.* Without loss of generality, I define positive and negative Ts to be as in Fig. 1; and the positive and negative Ds to be those developed from their respective solid. Table 1 clearly shows that in these matrices, GE:GE axial matings always occur in pairs, e.g. $\sqrt{1}$: T^+: $\{T^-, D^-\}$; $\sqrt{3}^{\alpha}_{\beta}$: T^+: $\{D^+, T^-\}$; and we shall see that this applies in general.

In the case of the GEs, the $\sqrt{2}$ diagonal edge elements of a polyhedron on each $\sqrt{1}$ axis alternate in orientation from top/side to bottom/opposite side; while the facial elements on each pair of coaxial α and β $\sqrt{3}$ axes change between triangles and hexagons, both of which have an associated orientation. This orientation is obvious in the case of the triangles (up- or down-ward pointing); in the case of the hexagons it is indicated in notation by the appendage of an extended triangle; for vertices, a small line.

Therefore, the potential proper relations of GE to GE are quite highly constrained. On the $\sqrt{1}$ axes, a GE can meet with one of its opposite sign, or with the other GE of opposite sign; but it cannot properly meet with either GE of its own sign. On the α and β $\sqrt{3}$ axes, a GE can only meet with one of its opposite sign, or with the other GE as the same sign as itself; but it cannot properly meet with itself, or with the other GE of the opposite sign. Figure 2 (again, at *http://rmeurant.com/its/hn2.html*) shows builds on each of the GEs.

These constraints are met in the solitary {2,3,3|2,3,3} honeycomb, which I describe in Part I as a four-way alternation, or mix-and-match. This gives four permutations, according to which GE associates with which reference tetrahedral lattice (refer Fig. 2), i.e.:

$$\begin{Vmatrix} D^- & T^- \\ D^+ & T^+ \end{Vmatrix}, \begin{Vmatrix} D^+ & D^- \\ T^+ & T^- \end{Vmatrix}, \begin{Vmatrix} T^+ & D^+ \\ T^- & D^- \end{Vmatrix}, \begin{Vmatrix} T^- & T^+ \\ D^- & D^+ \end{Vmatrix}.$$

4 Axial Relationships of GE: PP pairs

Now consider the potential relations between GEs and PPs. On $\sqrt{1}$ axes, GEs only develop transverse diagonal edges, which no PP does (for OH and TO, the transverse edge is on the $\sqrt{2}$ axis, and the $\sqrt{1}$ axis element of the TO is the rotated square, which is merely bounded by non-axial diagonal edges). The GEs do not develop symmetry on $\sqrt{2}$ axes; so we are only concerned with how GEs and PPs relate on $\sqrt{3}$ axes.

Again, the potential matings of polytopes are evidently constrained. On the α and β $\sqrt{3}$ axes, Ts only dance with CBs or VTS, by vertex; or SRs or OHs, by downward pointing triangle. Ds dance only with TCs and COs, by upward pointing triangle; or with GRs and TOs, by hexagon. Each GE can dance with one set of four of the PPs:

T: { $CB, VT; SR, OH$ } D: { $GR, TO; TC, CO$ }

Tables 2 and 3 show the pairings, and arrays. Note the arrowed expansion sequences.

These pairings are identical to those in Table 2 above for the $\sqrt{3}$ PP axes, and indeed correlate with the possible {2,3,3|2,3,4} arrays shown in Table 3, namely: $\begin{vmatrix} T^+ & VT \\ OH & T^- \end{vmatrix}$, $\begin{vmatrix} T^+ & CB \\ SR & T^- \end{vmatrix}$, $\begin{vmatrix} D^+ & CO \\ TO & D^- \end{vmatrix}$, and $\begin{vmatrix} D^+ & TC \\ GR & D^- \end{vmatrix}$, together with their permutations.

5 Axial Relationships of PP: PP pairs

By a similar procedure, I compare pairs of PP along the $\sqrt{1}, \sqrt{2}, \sqrt{3}$ axes. Their relationships can be determined on the basis of whether they are compatible or not, that is, whether they can mate together, at either a vertex, a (transverse or occasionally axial) edge, or an (axial or occasionally transverse) face. In the case of PP: PP, i.e. {2,3,4|2,3,4} symmetry arrays, this can be proximal, where they make actual contact (vertex, edge or face); but in some cases it can also be distal, through a secondary neutral intermediary (which could be an axial edge, neutral polygonal face, or neutral polyhedron - a prism).

This reveals a striking fact, and confirms the earlier behavior for the GE: GE and GE: PP matings; in each axial case, polytope matings - in this case PP: PP - form natural pairs, and these differ for each axis, with further complexity developing on the $\sqrt{3}$ axes, as in Fig. 3.

The pairs are: $\sqrt{1}\, pairs$: { $(GR, TC), (CB, SR), (TO, CO), (VT, OH)$ }

$\sqrt{2}\, pairs$: { $(GR, SR), (TC, CB), (TO, OH), (CO, VT)$ }

$\sqrt{3}\, self-reflective\ pairs$: { $(GR, TO), (CB, VT)$ }

$= \{ GR{:}GR, TO{:}TO, CB{:}CB, VT{:}VT, GR{:}TO, TO{:}GR, CB{:}VT, VT{:}CB \}$

$\sqrt{3}\, non-reflective\ pairs$: { $OH{:}CO, OH{:}TC, SR{:}CO, SR{:}TC$ }

$= \{ CO{:}(OH, SR), OH{:}(TC, CO), TC{:}(SR, OH), SR{:}(CO, TC) \}$

In Table 6, each pair correlates one-to-one with its corresponding {2,3,4|2,3,4} array.

In the case of $\sqrt{1}$ and $\sqrt{2}$ axes, each PP is self-reflective, so it mates with itself, as well as with just one other, its pair; but in the case of the $\sqrt{3}$ axes, note that four of the PPs are self-reflective, so each can mate with itself, while it can also mate with one other. But the other four PPs have triangular faces, which may alternate in orientation (point up or down); in these cases each PP cannot mate with itself, as the direction of apex flips between upper and lower. So for these four, in square array, each mates with its two neighbors, but not with its opposite, as in Fig. 3. In $\sqrt{3}$ matrices, common mating conditions, situated in overlapping squares, accord with the expansion/contraction sequences of arrays discussed in Part 1.

6 Conclusion

This paper has considered how $GE{:}GE$, $GE{:}PP$, and $PP{:}PP$ pairs can combine or mate with one another, proximally or distally, along $\sqrt{1}$, $\sqrt{2}$, or $\sqrt{3}$ axes, and how this relates to the honeycombs. The matings are highly constrained. $GE{:}GE$ matings correlate with the singular {2,3,3|2,3,3} array; $GE{:}PP$ matings correlate with the four {2,3,3|2,3,4} arrays; and $PP{:}PP$ matings correlate with the ten {2,3,4|2,3,4} arrays. In all cases, matings occur in pairs, and these pairs vary by axis, and by symmetry group; for any one symmetry group and axis, a constituent polyhedron pairs with just two others, and that association pattern is unique to the symmetry group and axis. In the case of the $PP{:}PP$ matings of the {2,3,4|2,3,4} symmetries, one of these will be with itself, except for $\sqrt{3}$ axis matings, which differentiate the PPs into two groups of four, which can be depicted in two square arrangements. $PP{:}PP$ pairings of the first group behave in a similar manner to $GE{:}GE$ and $GE{:}PP$ pairings, with PPs pairing with themselves and with their opposites; conversely, those of the second group do not; instead, each PP pairs with its two neighbors, but not with itself or its opposite. Furthermore, for the $GE{:}PP$ and $PP{:}PP$ pairings, the expansion/contraction sequences discussed in Part I are evident in the $\sqrt{3}$ matrices; more particularly, for the $PP{:}PP$ pairings the sequences are evident as overlapping squares in Table 4 (lower left) and Table 6.

We therefore move beyond mere recognition of sets of GEs and PPs, to an appreciation of a profound inner order that relates the individual elements, according to their potential for mating properly with one another, and according to the proper honeycombs that they form. This respects, but seeks to surpass, prior efforts to describe their overall structure [5, 6]. The challenge then is to evince an adequate formal representation of that profound harmony, which is the new order that this series of papers pursues, and which the third paper in this series will address.

Figures and Tables

For reasons of limited space, these are all provided at *http://rmeurant.com/its/hn2.html* .

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