A New 2.5D Schema of the Regular and Semi-regular Polyhedra and Tilings: Classes II and IV

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1. Abstract/Introduction

Inspired by Critchlow [1], I previously advanced a new order in space to account for the regularities and relationships of the regular and semi-regular polyhedra, in three classes I–III of {2,3,3}, {2,3,4}, and {2,3,5} symmetry, and its extension to the regular and semi-regular tilings of the plane, of two classes IV & V of {2,3,6} and {2,4,4} symmetry [2]. Each class consisted of 7 polytopes, together with a further enantiomorphic 8^{th} snub polytope, which functioned as a transition polytope. Within each class, 3D polyhedra (2D polygonal arrays) were presented in inverted "T" form, consisting of a horizontal base truncation sequence between positive (+ve) polar and negative (–ve) polar polytope: PL^+– TP^+– QR – TP^-– PL^-; and a vertical transcendent sequence: 1° QR – (1.5° SnbQR) – 2° SR – 3° GR (with SnbQR in either enantiomorph form). In this work, I assume that the reader is familiar with that earlier research, which PDF is downloadable from my homepage [2]. The recognition of the regular polyhedra of +ve & –ve tetrahedra, octahedron & cube, and icosahedron & dodecahedron as extreme polar forms, mediated by the perfect forms of the Class I–III quasi-regular polyhedra of the respective 'tetratetrahedron' (2-colored octahedron), 'octahexahedron' (cuboctahedron), and icosidodecahedron (and corresponding Class IV & V quasi-regular 2D polygonal tri–hex and square–square arrays), was a relevant critical insight informing another of my papers [3].

In this present paper, I advance a cubic 2.5-dimensional schema to better describe the morphological structure of each class, by firstly positing a null polytope VP; then transforming the structural order of each class from the inverted T form of two sequences, into a cubic schema of two expansion sequence clusters, whereby in either, a seed polytope expands in either of two ways into two kinds of intermediary polytope, then in the other way, to a fully developed polytope. I describe this in relation to the most important Class II of {2,3,4} symmetry, as it corresponds to the polyhedra that constitute all 16 (10 distinct) of the Class III honeycombs. To confirm the validity of the order, I also describe Class IV of {2,3,6} symmetry of the regular and semi-regular tilings of the plane, which like Class II & III, is asymmetric: polar opposites are not the same though recolored polytopes, but different polytopes. The diverse correspondences of the order across classes are rigorous. In a longer development of this paper, I anticipate describing all five classes of the revised new order in space, characterizing the polytopes and their expansion sequences and clusters thereof.

Keywords: all-space-filling, polyhedra, honeycomb, tiling, structural morphology, semi-regular, order.

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A New Order in Space [2]: available at **PDF 06** at http://www.rmeurant.com/its/papers/polygon-1.html

(NB, to conserve space, the cubic 2.5D schema in this paper is depicted throughout in cuboid form).

Nomenclature

+ve, positive; –ve, negative; C, class; CB, Cube; CO, Cuboctahedron; DD, Dodecagon (12-gon)*;* E, Edge; E^0, Neutral Edge; F: Face; GR, Great Rhombic; GRCO, Great Rhombic Cuboctahedron; HX, Hexagon; NS, Neutral Square; ntrl, Neutral; OG, Octagon; OH, Octahedron; P, Pole/Polar; $PL^{+/-}$, +/–ve Polar; QR, Quasi-regular; RS, Rotated (tRuncated) Square; RT, Rotated (tRuncated) Triangle; RX, Rotated Hexagon; Snb, Snub; SnbCO, Snub Cuboctahedron; SnbQR, Snub Quasi-regular; SQ, Square; SR, Small Rhombic; SRCO, Small Rhombic Cuboctahedron; T or Tr, Truncated; TC, Truncated Cube; TH, Tri–Hex (array); TO, Truncated Octahedron; TP, Truncated Polar; $TrP^{+/-}$, Truncated +/–ve Polar; TR, Triangle; $V^{+/-}$, +/–ve Vertex; V^0, Neutral Vertex; VP_C, null Vertex Polytope of Class C.

2. Honeycomb/Tiling Investigations

Inspired by Grünbaum and Shephard [4], I later investigated the structural morphology of the all-space-filling polyhedral honeycombs of Class I–III of {2,3,3|2,3,3}, {2,3,3|2,3,4}, and {2,3,4|2,3,4} symmetry, and the corresponding polygonal tilings of the plane of Classes IV & V of {2,3,6} & {2,4,4} symmetry [5–7]. I gained significant insight into the honeycomb order when I posited for each class C the null quasi-regular polytope, as the vertex of zero dimension, with zero extension, but with +ve and –ve polar vertices V^+ and V^-, and neutral (ntrl) vertices V^0, all coincident in the **verticial polytope VP_C** = $V^+ + V^0 + V^-$, while allowing spatial extension in the expansion sequences. This insight was first in relation to the various 3D honeycombs, especially the polyhedral honeycomb Class III of {2,3,4|2,3,4} symmetry, and later extended to the 2D honeycomb Class IV & V of {2,3,6} & {2,4,4} symmetry [5–7].

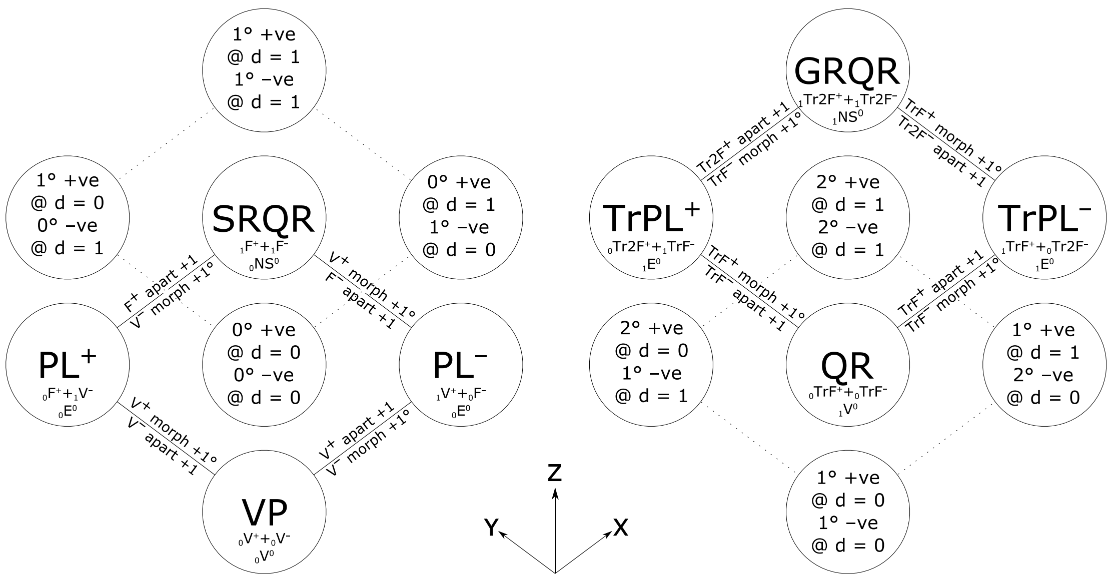

Fig. 1. Archetypal expansion sequence diamond clusters at lower (left) and upper (right). Texts of upper left and lower right diamonds refer to the faces of their lower and upper polytopes, respectively. 0° face is V, 1° face is PF (left); 1° face is TrPF, 2° is twice truncated = double frequency face (right). Seed polytope expands in 2 ways, as one pole morphs while other separates by +1, and *vice versa*; then morphed pole separates by +1 while separated pole morphs, and *vice versa*, to common rhombic QR.

Further progress was made through the organizing principle of the common formal structure of an expansion sequence cluster, of contracted seed honeycomb that in the first stage, expands in either of two ways to two intermediary honeycombs; and in the second stage, expands further in the other of the two ways, respectively, to the fully expanded honeycomb [5–7]. A common pattern to all 16 Class III {2,3,4|2,3,4} honeycombs (of 10 distinct forms) [6: fig. 2] then became evident. The schema was also seen in the Class IV & V honeycomb 2D tiling arrays, allowing its recognition in the Class IV & V polytope arrays, and abstraction to the Class I–III polytopes. This revised order and 2.5D schema of the polyhedra enabled deep insight into the structural morphology of the regular & semi-regular polytopes (Fig. 1).

3. The revised order of the Class II polytopes and their 2.5D schema

The initial step of introducing the null polytope, in this Class II, the null CO VP_{II}, is to locate it on the central transcendent vertical axis, below the QR CO, in effect a 0° element. But this does not provide adequate relation to the other polyhedra. The appropriate formal structure then becomes two overlapping diamonds separated vertically (omitting SnbCO), recognizing two groups of 4 of the now 8 elements, in which each element of one group has a unique pair in the other group, logically pairing the lower form with its upper truncated form (Figs. 2, 3).

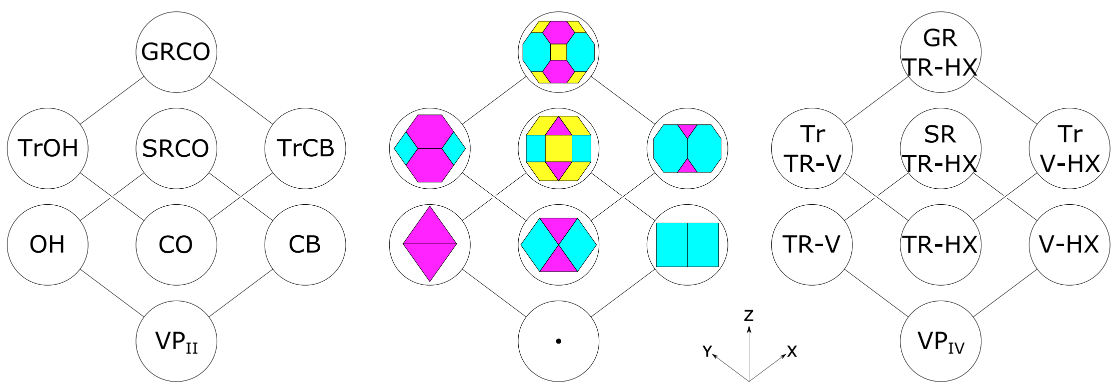

Fig. 2. Overlapping diamonds of the schema: Class II (left) & on neutral axis (center), Class IV (right).

This suggested investigating the corresponding transformations on each of the *x*, *y*, and *z* axes of the figure, where the two overlapping diamonds could now be recognized as the lower and upper faces of a cube in projection, in which those *x*, *y*, *z* axes could be examined (Fig. 3).

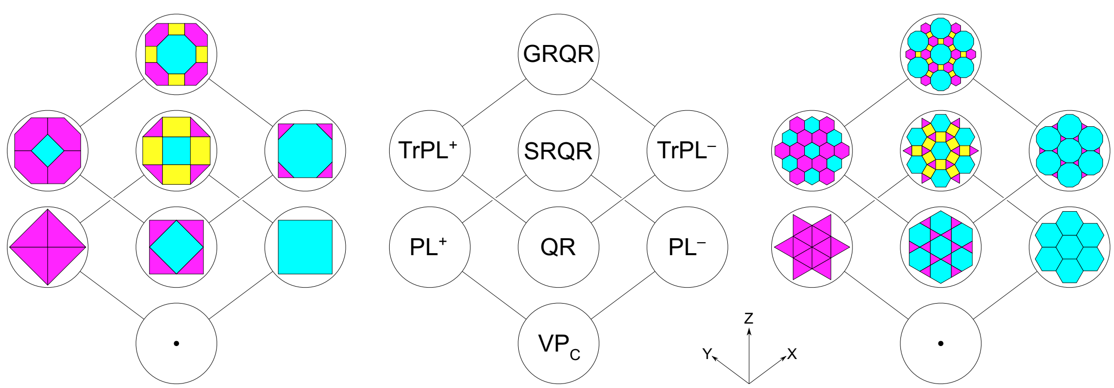

Fig. 3. Class II {2,3,4} (left), general 2.5D schema (center), Class IV {2,3,6} (right), on negative axis.

This schema shows a high degree of regularity. As previously mentioned, the vertical axis transformations correspond to truncations, hence the null polytope VP_{II} can be deduced to be a truncation of the QR CO; V^+, V^0, & V^- coincide in the vertex VP_{II}: $VP = V^+ + V^0 + V^-$.

In the lower diamond sequence cluster, the seed form of the null VP_{II} expands on the *x*-axis to the PL^- CB, and *y*-axis to the PL^+ OH: $^{OH} \nwarrow _{NV} \nearrow ^{CB}$; then on the *x*-axis the OH, and on the *y*-axis the CB, expand to the common neutral SRCO: $_{OH} \nearrow ^{SRCO} \nwarrow _{CB}$.

In the upper diamond sequence cluster, the seed form of the neutral QR CO expands on the *x*-axis to the TP^- TC, and *y*-axis to the TP^+ TO: $^{TO} \nwarrow _{CO} \nearrow ^{TC}$; then on the *x*-axis the TC, and on the *y*-axis the TO, expand to the common neutral GRCO: $_{TO} \nearrow ^{GRCO} \nwarrow _{TC}$, correspondingly.

Polyhedra axes are common to all polyhedra in a class, but vary class-to-class. Class II –ve, ntrl, & +ve axes are defined as the normal axes of the (100), (110), & (111) planes, respectively. OH vertices and CB faces are –ve, the edges of both in two orientations are ntrl, while OH TR faces and CB vertices are +ve. The QR CO is composed of –ve RSs and +ve RTs. Transformations of the constituent polytopes (faces/edges/vertices) of the 8 polyhedra or polygonal arrays in a class can then be characterized by axis of the 2.5D schema. For Class II:
<u>On the *z*-axis</u> (↿), –ve elements increase 1°: ($V_0^- \upharpoonleft RS_0^-, SQ_0^- \upharpoonleft OG_0^-, V_1^- \upharpoonleft RS_1^-, SQ_1^- \upharpoonleft OG_1^-$); ntrl elements separate, so adjoining vertices become unit distance apart: ($V_0^0 \upharpoonleft V_1^0$, $2\times(E_0^0 \upharpoonleft E_1^0)$, $NS_0^0 \upharpoonleft NS_1^0$), the $2\times(E_0^0 \upharpoonleft E_1^0)$, being in 2 sets of different orientations; and +ve elements increase 1°: ($V_0^+ \upharpoonleft RT_0^+, TR_0^+ \upharpoonleft HX_0^+, V_1^+ \upharpoonleft RT_1^+, TR_1^+ \upharpoonleft HX_1^+$).
<u>On the *x*-axis</u> (↗), –ve elements increase 1°: ($V_0^- \nearrow SQ_0^-, RS_0^- \nearrow OG_0^-, V_1^- \nearrow SQ_1^-, RS/_1^- \nearrow OG_1^-$); N elements increase 1°: $2\times(V^0 \nearrow E^0$, $E^0 \nearrow NS)$; and +ve elements separate, so adjoining vertices become unit distance apart: ($V_0^+ \nearrow V_1^+, TR_0^+ \nearrow TR_1^+, RT_0^+ \nearrow RT_1^+, HX_0^+ \nearrow HX_1^+$).
<u>On the *y*-axis</u> (↖), *reading backwards*, –ve elements separate, so adjoining vertices become unit distance apart: ($V_1^- \nwarrow V_0^-, SQ_1^- \nwarrow SQ_0^-, RS_1^- \nwarrow RS_0^-, OG_1^- \nwarrow OG_0^-$); ntrl elements increase 1°: ($E_0^0 \nwarrow V_0^0, NS_0^0 \nwarrow E_0^0, E_1^0 \nwarrow V_1^0, NS_1^0 \nwarrow E_1^0$); and +ve elements increase 1°: $TR_0^+ \nwarrow V_0^+, HX_0^+ \nwarrow RT_0^+, TR_1^+ \nwarrow V_1^+, HX_1^+ \nwarrow RT_1^+$) (refer Figs. 1–3).

The Class IV –ve axes are defined as the normal axes of the mid-faces of the triangles of the PL TR array, the ntrl axes in two sets of orientations as the normal axes of the mid-edges of the PL TR and PL HX arrays, respectively, and the +ve axes as the normal axes of the mid-faces of the hexagons of the PL HX array, respectively. Hence the PL^+ TR array V^- and PL^- HX array HX are –ve, the edges of both polar arrays in two sets are ntrl, while the PL^+ TR array TR and PL^- HX array V^- are +ve. The QR TR–HX (TH) array is composed of –ve RXs and +ve RTs. Transformations of the constituent polytopes (faces/edges/vertices) of the 8 Class IV polygonal arrays can then be characterized by axis of the 2.5D schema:
On <u>the z-axis (↿)</u>, –ve elements increase by 1°: ($V_0^- \upharpoonleft RX_0^-, HX_0^- \upharpoonleft DD_0^-, V_1^- \upharpoonleft RX_1^-, HX_1^- \upharpoonleft DD_1^-$); ntrl elements separate, so that adjoining vertices became unit distance apart: ($V_0^0 \upharpoonleft V_1^0$, $2\times(E_0^0 \upharpoonleft E_1^0)$, $NS_0^0 \upharpoonleft NS_1^0$), the $2\times(E_0^0 \upharpoonleft E_1^0)$ being in 2 sets of different orientations; while +ve elements increase by 1°: ($V_0^+ \upharpoonleft RT_0^+, TR_0^+ \upharpoonleft RX_0^+, V_1^+ \upharpoonleft RT_1^+, TR_1^+ \upharpoonleft RX_1^+$).
<u>On the x-axis (↗)</u>, –ve elements increase by 1°: ($V_0^- \nearrow HX_0^-, RX_0^- \nearrow DD_0^-, V_1^- \nearrow HX_1^-, RX_1^- \nearrow DD_1^-$); ntrl elements increase by 1°: ($V_0^0 \nearrow E_0^0, E_0^0 \nearrow NS_0^0, V_1^0 \nearrow E_1^0, E_1^0 \nearrow NS_1^0$); and +ve elements separate, so that adjoining vertices become unit distance apart: ($V_0^+ \nearrow V_1^+, TR_0^+ \nearrow TR_1^+, RT_0^+ \nearrow RT_1^+, RX_0^+ \nearrow RX_1^+$).
<u>On the y-axis (↖)</u>, *reading backwards*, –ve elements separate, so adjoining vertices become unit distance apart: ($V_1^- \nwarrow V_0^-, HX_1^- \nwarrow HX_0^-, RX_1^- \nwarrow RX_0^-, DD_1^- \nwarrow DD_0^-$), ntrl elements increase by 1°: ($E_0^0 \nwarrow V_0^0, NS_0^0 \nwarrow E_0^0, E_1^0 \nwarrow V_1^0, NS_1^0 \nwarrow E_1^0$); and +ve elements increase 1°: ($TR_0^+ \nwarrow V_0^+, RX_0^+ \nwarrow RT_0^+, TR_1^+ \nwarrow V_1^+, RX_1^+ \nwarrow RT_1^+$). (*NB*, all 2D polytope arrays (Figs. 2 & 3) repeat to infinity).

The snub form is clearly a transitional form, as evidenced in the jitterbug sequences I recognized in my early research [2] for each polytope Class I–V contraction SRQR → SnbQR → QR; in Class II, SRCO → SnbCO → CO; and in Class IV, SRTH → SnbTH → TH. The Snub polytope in either or both of its enantiomorphs thus sits at the center of the schematic 2.5D cube its class, mediating the lower and higher diamonds. Class II SnbCO or Class IV SnbTH is, as before, between the 1° QR CO or TH and the 2° SRCO or SRTH, respectively.

The original linear horizontal truncation polytope sequence of PL^+ – TP^+ – QR – TP^- – PL^- is devolved in the 2.5D schema to a revised 'M'-shaped sequence; while the original neutral linear vertical transcendent sequence now extends to include the null verticial polytope VP.

4. Conclusion

This 2.5D schema and revised order of the regular and semi-regular 3D polyhedra and 2D polygonal arrays proves vital towards characterizing and appreciating their structural morphology, and is applicable across classes; the behavior in one class is perfectly reflected in the other classes, as in the exemplar jitterbug sequences. In addition, the schema recognizes for each class an additional null quasi-regular polytope VP_C, whose virtual properties are deducible from the regularities of the schema. The schema exploits the same motif of diamond expansion sequence cluster derived from the Class III all-space-filling polyhedral honeycombs and Class IV & V polygonal tiling patterns of the plane, so appears archetypal.

In future research, I anticipate developing this paper to include all five classes I–V of the regular and semi-regular 3D polyhedra and 2D polygonal tilings to present a complete revised new order in space; and in later work, reapply the schema and the component diamond to the five classes I–V of the all-space-filling honeycombs and tilings, in order to gain further insight into their structural morphology, with potential applications to lattice structure, bone scaffolding, polymer composites, kinetic space structures, and in general, to appreciate and appropriately utilize the inherent subtle and profound harmonic structure of empirical space.

5. References

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