Form and Counterform in the All Space-filling Periodic Polyhedral Honeycombs

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Abstract

This third part of a series of papers investigates the ordering of the all-space-filling periodic polyhedral honeycombs from the perspective of form and counterform. Each periodic polyhedral honeycomb can be differentiated into a subset of contiguous periodic polyyopes, the *form*, while the leftover space constitutes another subset of contiguous periodic polytopes, the *counterform*. Together, they fill all space. In certain cases, the contiguity of a subset is through just neutral vertex; axial, transverse, or diagonal edge; or diagonal prism, rather than through neutral axial polygon or polyhedron; but the behavior is rigorous, and further confirms the legitimacy of the meta-order of these honeycombs that I have previously advanced. Form and counterform are interchangeable, depending on which is being attended to, as in figure-background perception in psychology. Nevertheless, each demonstrates parallel and consistent structure within each of the three symmetry classes. In all cases, form and counterform are intertwined with each other, and divide space into two. In just four cases, form and counterform are identical, and in another, identical but enantiomorphic. The expansion sequences of honeycombs in Classes II and III that I elsewhere identify apply, so that in each sequence, its form exhibits an expansion sequence, whilst its counterform simultaneously exhibits a corresponding sequence; these are consistent for each sequence in the class.

Keywords: all-space-filling, polyhedra, honeycomb, structural morphology, form, counterform, order.

Supplementary Information: Graphics can be found at http://www.rmeurant.com/its/SI-3.html

Glossary

GE : The 4 Great Enablers, of +/– orientation Tetrahedron T^+, T^-, or truncateD tetrahedron, D^+, D^-.

PP : The 8 Primary Polytopes: VerTex VT, CuBe CB, Truncated Octahedron TO, Great Rhomic cuboctahedron GR, OctaHedron OH, CubOctahedron CO, Truncated Cube TC, and Small Rhombic octahedron SR.

NE : The 10 Neutral Elements: the Diagonal Edge DE (i.e., √2), (2D) Neutral Vertex NV, (transverse) Square SQ, Rotated Square RS, and OctaGon OG, and their respective prisms, the Diagonal Prism DP (axial square), Axial Edge AE, Square Prism SP (neutral cube), Rotated Prism RP (rotated cube), and Octagonal Prism OP.

RCL : Reference Cubic Lattice, RCL_1 and RCL_2, each with nodes at the centers of the other's cubes.

RTL : Reference Tetrahedral Lattice, there being two for each RCL: RTL_1^α and RTL_1^β, RTL_2^α and RTL_2^β.

$V3$: √3 axial Vertex "face" of VT, CB, or T. $V4$: √1 axial Vertex "face" of VT

TR : √3 axial Triangular face, +ve (upper Δ) TR^+ of OH, SR, or D; –ve (up. ∇) TR^- of TC, CO, or T.

HG : √3 axial Hexagonal face, of TO, GR, or D.

1. Introduction

This third part of a series of papers investigates the ordering of the all-space-filling periodic polyhedral honeycombs that I have previously advanced [1–3], from the perspective of form and counterform, which together, fill all space. This paper assumes that the reader is familiar with that earlier research. My analysis of the polyhedral honeycombs differentiates them into three classes on the basis of the symmetry their lattices display, viz. Classes I {2,3,3|2,3,3} (1 kind); II: {2,3,3|2,3,4} (4 distinct kinds); and III: {2,3,4|2,3,4} (10 distinct kinds). These honeycombs consist of various combinations of the two "Great Enablers" $GE = T$ and D in positive or negative orientation (T^{+}, T^{-}, D^{+}, D^{-}), 8 Primary Polytopes (VT, CB, TO, GR, OH, CO, TC, SR), and Neutral Elements that separate them along the √1 XYZ axes. GEs and PPs are situated at the nodes of two reference cubic lattices RCL_1 and RCL_2, the nodes RC_1 of one lattice being located at the nodes RC_2 that are the centers of the cubes of the other lattice; and either RCL can be differentiated into two component tetragonal lattices RTL^{α} and RTL^{β}, so RTL_1: RTL_1^{α} of nodes RT_1^{α}, and RTL_1^{β} of RT_1^{β}; and RTL_2: RTL_2^{α} of nodes RT_2^{α}, and RTL_2^{β} of RT_2^{β}. The three classes are characterized by different types of alternation, which are engendered by whether GEs or PPs are situated at the nodes of the Reference RCLs or RTLs.

In reverse order, Class III is characterized by a simple alternation of PP_1 and PP_2 on RCL_1 and RCL_2, though firstly, both RCLs could be distinguished into their two respective component RTLs, RTL^{α} and RTL^{β}, to constitute an alternation of alternations. Secondly, the Class III honeycombs can be further differentiated into whether they are self-reflective ($PP_1 = PP_2$) (i.e. the same polyhedron), or not ($PP_1 \neq PP_2$). Class II is characterized by an alternation of alternations, as in RCL_1 consisting of an alternation of PP_1 and PP_2, while RCL_2 consists of an alternation of GE^{+} and GE^{-}, where for the one honeycomb, GE is either $T^{+/-}$ or $D^{+/-}$, but cannot be both. Finally, Class I is characterized by a complex four-way alternation of both plus and minus forms of both GEs, so that RCL_1 = [$RTL_1^{\alpha} = T^{+}$ and $RTL_1^{\beta} = D^{-}$], while RCL_2 = [$RTL_2^{\beta} = D^{+}$ and $RTL_2^{\beta} = T^{-}$], where superscripts +/– denote alternative orientations (of T or D in its reference circum-cube). Neutral Elements (NEs) of {2,2,4} symmetry are located on the respective √1 XYZ axes, so demonstrate one of 3 primary XYZ orientations.

2. The three classes of honeycombs and their constituent sets of polytopes

The Class III honeycombs are constituted of either 1 or 2 PPs and their respective NEs, and comprise four different two-step expansion/contraction sequences of honeycomb from contracted form through intermediary form to expanded form. These consist of one primary sequence, two secondary sequences that are enantiomorphs of one another, and one tertiary sequence. Steps in each sequence are characterized by firstly, one set of PPs separating by unit distance, while the other set morphs from one PP to another in the first step; and secondly, the first set of PPs then morphing from one PP to another, while the other (new) set of PPs separates by unit distance. There are only 4 kinds of morphs. (This lattice expansion sequence is well described in the earlier papers [3]). So in the various Class III honeycombs, pairs of proximate PPs can either be contiguous (*adjoining*; in the contracted state in the contracted or intermediary forms; or non-contiguous, separated by unit distance (*adjacent*; in the expanded state in the intermediary or expanded forms of the honeycombs. Meanwhile, in the contracted state of PPs, NEs can only have zero extent along their primary X, Y, or Z

(XYZ) axis, so are considered to be either neutral vertex (NV), transverse (√1) (TE) or diagonal (√2) (DE) edge, or transverse polygon of neutral square (NS), rotated square (RS), or octagon (OG). In the expanded state of PPs, these neutral polytopes have been projected along their primary axis (by unit length), to form axial edge (AE), transverse (√1) square SQ or diagonal (√2) prism (DP), or prismatic (neutral) square prism (neutral cube) (SP), rotated square prism (rotated cube) (RP), or (regular) octagonal prism (OP). Note again that each of these neutral polytopes has a primary XYZ axis, and two minor XYZ axes, so they come in three primary orientations. For practical applications, they can each be further differentiated if needed into +ve or –ve forms, according to the direction of their normal along the XYZ axis.

The Class II honeycombs are constituted of one or other of the GEs in both +ve and –ve orientations and 2 PPs that share the same √1 faces, and comprise four different one-step expansion/contraction sequences of honeycomb from contracted form to expanded form, and their respective NEs: NE_{GE} = NE_T or NE_D, and NE_{PP}. These (sequences) consist of two parallel one-step sequences, one for GE=$T^{+/-}$, and one for GE=$D^{+/-}$. Somewhat akin to Class III, the step in either sequence is characterized by the set of GEs separating by unit distance, while the two PPs of the other set morphs to the two other PPs. The four morphs are the same as for Class III. In a sequence, one reflective PP and one non-reflective PP of the contracted honeycomb morph to another reflective PP and another non-reflective PP, respectively, of the expanded honeycomb. GEs in the contracted honeycomb are contiguous, though mediated by the NE of DE; while the two PPs are contiguous, though mediated by the NE of NV or RS. In the expansion, the GEs separate by unit distance, and the neutral DEs project along their primary XYZ axis to become DP; while the two PPs morph to two other PPs, their NEs expanding from NV to NS or RS to OG.

The Class I honeycomb is constituted of both GEs in both +ve and –ve orientations, and their respective NEs = DP. It thus shows no expansion/contraction sequences of honeycomb from contracted form to expanded form. Along the axes of RCL_1, D^+ and T^- alternate, and are contiguous, though mediated by their NE of DE transverse to their primary XYZ axis, of alternating orientation; while along RCL_2, D^- and T^+ alternate, and are contiguous, though mediated by their NE of DE transverse to their primary XYZ axis, of alternating orientation.

This analysis means that the constituent polytopes of any of these honeycombs may be differentiated into subsets of polytopes, as follows: Class III: one or two PP, and their respective NEs; Class II: one or other GE in both orientations GE^+ and GE^- and 2 PPs, and their respective NEs (which are at maximum, 2D); and Class I: characterized by 2 (both) GEs in both +ve and –ve orientations, i.e. $RT_1^\alpha = D_1^+$, $RT_1^\beta = T_1^-$, $RT_2^\alpha = T_2^+$, $RT_2^\beta = D_2^-$; and their respective $NE_{GE}^{+/-}$s. The NE can be divided into NE_1 and NE_2, which can be differentiated as: NE_1^X, NE_1^Y, NE_1^Z, and NE_2^X, NE_2^Y, NE_2^Z; and as +ve or –ve according to their axial direction.

Class II is characterized by $RT_1^\alpha = T_1^+$ or D_1^+, $RT_1^\beta = T_1^-$ or D_1^-, $RT_2^\alpha = PP_2^+$, $RT_2^\beta = PP_2^-$, where for a specific honeycomb, “or” is exclusive; and PP^+ and PP^- are NOT the same polyhedron, but are rather √1 complementary – the two PPs that exhibit the same √1 face.

Class III can be rectified to a similar 4-fold alternation of tetrahedral lattices by considering the colorings of both PPs, where the “coloring” differentiates +/– pairs of either PP according to their α/β location, so that Class III is characterized by $RT_1^\alpha = PP_1^+$, $RT_1^\beta = PP_1^-$, $RT_2^\alpha = PP_2^+$, $RT_2^\beta = PP_2^-$. In the four self-reflective Class III honeycombs, $PP_1 = PP_2$

(i.e. all 4 polyhedra are the same polyhedron, but in different locations); in the six non-self-reflective honeycombs, $PP_1 \neq PP_2$ (where the two polyhedra are instead √3 complementary, sharing the same, 180° rotated √3 face: OH:CO/TC; CO:SR/OH; TC:SR/OH; SR:CO/TC).

So Class I may be considered as the four-fold alternation of 4 interpenetrating tetrahedral lattices of both GEs of both orientations; Class II as the alternation of alternation of two pairs of interpenetrating tetrahedral lattices, one pair being GEs and the other pair being √1 complementary PPs VT:OH, CO:TO; CB:SR; TC:GR; and Class III the alternation of alternation of two pairs of interpenetrating tetrahedral lattices, where either pair consists of the same PP; but the two PPs of RCL_1 and RCL_2 are √3 complementary PPs, which in the contacted and expanded honeycombs of the primary and tertiary sequences are the same PP, but in the secondary sequence and the intermediary forms of the other two sequences differ.

3. Form and Counterform Arrays

Having deposed the polyhedra of the periodic honeycombs into classes and into the constituent sets and locations, and identified neutral polytopes that can be considered to separate the GEs and PPs, the arrays of combinations of these constituent sets can be addressed. Of course any combination of the constituent sets of polyhedra and neutral polytopes of a honeycomb, or more generally its class, can be considered, and for Classes II and III, in regard to its sequence – and abstracted across sequences. But the present enquiry is concerned with form and counterform. In both cases, these are characterized by contiguity, even though that contiguity might only be through mediating VT, TE, DE, AE, or DP; and together, form and counterform fill all space, so that they consist of interpenetrating arrays sharing a common surface. Practical applications might well consider configurations where one or other are not contiguous; a contiguous form might serve as the reticulated provision of services and service spaces, while the counterform might be discrete (non-contiguous) usable, even habitable spaces – or at the organic level, artificial bone tissue scaffolding and discrete pores. Or it might consider three or more interpenetrating arrays. But in this paper, only the all-space-filling combination of just two arrays will be addressed. These reveal common structure over the three Classes, and are identified according to axes of contiguity, and class.

3.1. √1 XYZ Axes of Contiguity

3.1.1 √1 Axes of Contiguity for Class III.

The inspiration for the exploration of form and counterform came from the cubic array formed by core cubes in face-to-face contact with intermediary cubes, so that each core cube is in face-to-face contact with six intermediary cubes, while each intermediary cube is in face-to-face contact with two core cubes, one at each end. This might be regarded as an archetypal form; the leftover space, the "exterior", proves to be precisely the same array, though displaced. So the common surface of squares separates two arrays that here are identical. Space is divided into two interpenetrating compartments, the exterior of one being the interior of the other. One is form, the other counterform. The meta-order of honeycombs I advance subsumes this as a division of the $[CB_1 \mid CB_2]_{SP_1}^{SP_2}$ expanded array of the primary sequence in Class III, with CB_1 and its neutral SP_1 as form, and CB_2 and its neutral SP_2 as counterform.

In general, form and counterform (CNTR) for the Class III √1 axes of contiguity can be:

FORM | CNTR: <PP_1 : NE_1 : PP_1> | <PP_2 : NE_2 : PP_2>. Hence, Table 1:

Table 1. Class III √1 Honeycombs and Contiguous Form and Counterform Arrays.

Honeycomb	FORM array	CNTR array	Sequence	EXP
$[GR_1 \mid GR_2]_{OP_1}^{OP_2}$	$<GR_1 : OP_1 : GR_1>$	$<GR_2 : OP_2 : GR_2>$	Tertiary	2
$[TO_1 \mid GR_2]_{RP_1}^{OG_2}$	$<TO_1 : RP_1 : TO_1>$	$<GR_2 : OG_2 : GR_2>$		1
$[TO_1 \mid TO_2]_{RS_1}^{RS_2}$	$<TO_1 : RS_1 : TO_1>$	$<TO_2 : RS_2 : TO_2>$		0
$[SR_1 \mid TC_2]_{OP_1}^{OP_2}$	$<SR_1 : OP_1 : SR_1>$	$<TC_2 : OP_2 : TC_2>$	Secondary	2
$[OH_1 \mid TC_2]_{AE_1}^{OG_2}$	$<OH_1 : AE_1 : OH_1>$	$<TC_2 : OG_2 : TC_2>$		1
$[SR_1 \mid CO_2]_{SQ_1}^{RP_2}$	$<SR_1 : SQ_1 : SR_1>$	$<CO_2 : RP_2 : CO_2>$		1
$[OH_1 \mid CO_2]_{VT_1}^{RS_2}$	$<OH_1 : VT_1 : OH_1>$	$<CO_2 : RS_2 : CO_2>$		0
$[CB_1 \mid CB_2]_{SP_1}^{SP_2}$	$<CB_1 : SP_1 : CB_1>$	$<CB_2 : SP_2 : CB_2>$	Primary	2
$[VT_1 \mid CB_2]_{AE_1}^{SQ_2}$	$<VT_1 : AE_1 : VT_1>$	$<CB_2 : SQ_2 : CB_2>$		1
$[VT_1 \mid VT_2]_{NV_1}^{NV_2}$	$<VT_1 : NV_1 : VT_1>$	$<VT_2 : NV_2 : VT_2>$		0

Notes: Exp: Degree of Expansion: 2: Expanded; 1: Intermediary; 0: Contracted.

3.1.2 √1 Axes of Contiguity for Class II.

In general, the form and counterform for the √1 Axes of Contiguity for Class II can be:

FORM | CNTR: $<GE_1^+ : NE_{GE} : GE_1^-> \mid <PP_2^+ : NE_{PP} : PP_2^->$

where, PP_2^+ and PP_2^- are complementary PPs pairs sharing the same √1 face. Hence, Table 2:

Table 2. Class II √1 Honeycombs and Contiguous Form and Counterform Arrays.

Honeycomb	FORM array	CNTR array	FORM array	CNTR array	Honeycomb
$\begin{vmatrix} T^+ & CB \\ SR & T^- \end{vmatrix}$	$<T_1^+ : DE : T_1^->$	$<CB_2^+ : SQ_2 : SR_2^->$	$<D_1^+ : DE : D_1^->$	$<TC_2^+ : RS_2 : GR_2^->$	$\begin{vmatrix} D^+ & TC \\ GR & D^- \end{vmatrix}$
$\begin{vmatrix} T^+ & VT \\ OH & T^- \end{vmatrix}$	$<T_1^+ : DE : T_1^->$	$<VT_2^+ : NV_2 : OH_2^->$	$<D_1^+ : DE : D_1^->$	$<CO_2^+ : RS_2 : TO_2^->$	$\begin{vmatrix} D^+ & CO \\ TO & D^- \end{vmatrix}$

3.1.3 √1 Axes of Contiguity for Class I.

In general, the form and counterform for the √1 Axes of Contiguity for Class I can be:

FORM | CNTR: $<GE_1^+ : NE_1 : GE_1^-> \mid <GE_2^+ : NE_2 : GE_2^->$

i.e., FORM | CNTR: $<D_1^+ : DE_1 : T_1^-> \mid <T_2^+ : DE_2 : D_2^->$

where, D_1^+ and T_1^- alternate to form chains of GEs along the √1 XYZ axes of RCL_1, while, T_2^+ and D_2^- alternate to form chains of GEs along the √1 XYZ axes of RCL_2.

3.2 √2 XYZ Axes of Contiguity

I do not here consider the √2 XYZ Axes of Contiguity form and counterform arrays.

3.3 √3 XYZ Axes of Contiguity

Note that these are dealt with in different order.

3.3.1 √3 Axes of Contiguity for Class I.

In the Class I honeycomb, D^+ on its triangular faces only mates with T^+ (the reduced size of the parent T it is truncated from), while D^- on its triangular faces only mates with T^-, to form interpenetrating arrays of $<D_1^+ - T_2^+>$ and $<D_2^+ - T_1^->$. Therefore:

FORM | CNTR: $<D_1^+ : TR_{12} : T_2^+> \mid <D_2^+ : TR_{21} : T_1^->$

This is an interesting pair of arrays, as either might be considered to consist of the array of 3-frequency T^+/T^- formed by D^+/D^- and T^+/T^-, which $3f\,T^+/T^-$ overlap at common T^+/T^-. Alternatively, D^+ on its hexagonal faces only mates with D^-, while T_2^+ on its vertex faces (its vertices) only mates with T_1^- to form interpenetrating arrays of $D^{+/-}$, and of $T^{+/-}$. Therefore:

FORM | CNTR: $<D_1^+ : HG_{12} : D_2^+>$ | $<T_2^+ : HG_{21} : T_1^->$

3.3.2 √3 Axes of Contiguity for Class II.

In Class II, PP alternates along its √3 axes in the order $(PP^+ - GE^+ - PP^- - GE^-)$ = $(VT^{\alpha}: NV^+: T^+: TR^+: OH^-: TR^-: T^-: NV^-)$ + $(CO^{\alpha}: TR^+: D^+: HG^+: TO^-: HG^-: D^-: TR^-)$, $(CB^{\alpha}: NV^+: T^+: TR^+: SR^-: TR^-: T^-: NV^-)$ + $(TC^+: TR^+: D^+: HG^+: GR^-: HG^-: D^-: TR^-)$, for the contracted, and the expanded forms, respectively (where the +/– designations of the NEs are not rigorous, but indicative only). Form and counterform can be obtained by associating GE^+ with PP^+, and GE^- with PP^-, or alternatively, GE^+ with PP^-, and GE^- with PP^+:

FORM | CNTR: $<GE_1^+ - PP_2^+>$ | $<GE_1^- - PP_2^->$, or, $<GE_1^+ - PP_2^->$ | $<GE_1^- - PP_2^+>$

PP_2^+ and PP_2^- are not the same polyhedra, but complementary PP pairs with the same √1 face.

3.3.3 √3 Axes of Contiguity for Class III.

Class III honeycombs are realized in similar manner as employed in Class II, by actualizing the coloring of PPs, so PP_1=PP_1^+ + PP_1^- and PP_2=PP_2^+ + PP_2^-. Disregarding √1 NEs:

FORM | CNTR: $<PP_1^+ - PP_2^+>$ | $<PP_1^- - PP_2^->$ or $<PP_1^+ - PP_2^->$ | $<PP_1^- - PP_2^+>$

depending on which tetrahedral array of PP_1 is mated with which tetrahedral array of PP_2. This works without practical concern for the honeycombs that do not have 3D NE by neglecting the √1 NEs, but needs to be taken account of in the honeycombs that do. This presents a formal problem, as it is not possible while maintaining true 3D symmetry.

4. Conclusion

This paper describes the relationships that characterize the fundamental structure of these honeycombs, individually, in sequence, and as various instances of the same meta-order. This meta-order embodies the fundamental structure of empirical 3-D space. These descriptions allow the configurations to become conceivable, imageable, and practical. Form and counterform of the honeycombs enable the structuring of interpenetrating but distinct spaces that could be engineered to provide controllable porous membrane surfaces that allow two domains to interact with one another through high surface area interfaces. Their geometry becomes accessible to diverse applications, e.g. tensile arrays in the marine environment and in Space (stressed by pneumatic envelope); chemical structures of composite and hybrid compounds; new materials engineered at nanoscale; filters; artificial bone tissue scaffolding.

5. References

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