Sequences of the All Space-filling Periodic Polyhedral Honeycombs

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Abstract

I have previously classified the proper honeycombs into three symmetry groups, identified the component polytopes of GE polyhedra, PP polytopes, and NE polytopes, and advanced a schema that adequately represents the relationship between the GEs and PPs and their corresponding honeycombs [1, 2]. Herein, I briefly describe the expansion sequences that relate honeycomb to honeycomb, where complementary sets of PPs separate or morph by turn, while their respective NEs stretch or expand.

Keywords: polyhedra, honeycomb, array, tessellation, spatial harmony, form, order

1. Introduction: Symmetry Classes of the All Space-filing Periodic Polyhedral Arrays

The all-space-filling periodic polyhedral honeycombs exhibit three symmetry classes of $\{2,3,4|2,3,4\}$, $\{2,3,3|2,3,4\}$, and $\{2,3,3|2,3,3\}$. The polytope components of these honeycombs are the 4 Great Enablers (GEs): (T^+, T^-, D^+, D^-); 8 Primary Polytopes (PPs) in two distinct classes of (VT, CB, TO, GR), which can self-reflexively form honeycombs, and (OH, CO, TC, SR), which do not; and Neutral Elements (NEs): (DE, NV, SQ, RS, OG) and their prisms (DP, AE, SP, RP, OP), which include the 2D NV and the non-semiregular OP.

The 10 distinct $\{2,3,4|2,3,4\}$ honeycombs are characterized by the alternation of two PPs, each situated at the nodes of a Reference Cubic Lattice (RCL), the nodes of one lattice being at the centers of the cubes of the other lattice. In self-reflexive honeycombs, the two PPs are the same polytope. In both kinds of honeycomb, NEs mediate adjoining pairs of PPs of the same kind along XYZ axes. In the simple lattices, these NEs are developed through the expansion sequences into 1D, 2D or 3D polytopes. In the bicubic 3D schema of the honeycomb order that I have previously advanced [1, 2], the two PPs are diagonal opposites on either the lower or upper squares, or are self-reflexive nodes of the upper square. Each of these 10 honeycombs might be described as a *simple alternation*.

The 4 distinct $\{2,3,3|2,3,4\}$ honeycombs are characterized by the same alternation of polyhedra at nodes of the two $RCLs$; but have a further alternation that develops of the two sets of tetrahedral nodes of either lattice. One RCL alternates nodes of one kind of GE of alternating orientation, +ve and –ve, so $RCL^1 = GE_\alpha^+ + GE_\beta^-$, while the other primary cubic lattice alternates nodes of different PPs: $RCL^2 = PP_1 + PP_2$. NEs are of 0D, 1D or 2D, and do not develop into 3D. In my 3D schema, the GEs are located in the central square, while the two different PPs are at opposite nodes of the long diagonals of the schema [2]. Each of these 4 honeycombs might be described as an *alternation of alternations*.

The distinct {2,3,3|2,3,3} honeycomb is characterized by a more complex alternation pattern, where one RCL alternates tetrahedral sets of T^+ and D^-, whilst the other RCL alternates tetrahedral sets of D^+ and T^-: $RCL^1 = T^+_\alpha + D^-_\beta$, and $RCL^2 = D^+_\alpha + T^-_\beta$. This shows a *true 4-way/fold alternation*. I recognize $10 + 4 + 1 = 15$ distinct honeycombs in all.

2. Expansion Sequences of the Regular Honeycombs

The {2,3,4|2,3,4} and {2,3,3|2,3,4} honeycombs may then be characterized into expansion sequences. The {2,3,4|2,3,4} honeycombs may be differentiated into one Primary, two Secondary, and one Tertiary sequences, each of two steps, i.e. 1. Contracted → Intermediate, and 2. Intermediate → Expanded. In the first step, the initial Contracted honeycomb can expand in two different ways, into two different Intermediate honeycombs of the same lattice size, depending on which set of PPs separates, and which set of PPs morphs. In the second step, the complementary process of separation and morphing occurs, so that both pathways culminate in the same final Expanded honeycomb. In the Primary and Tertiary sequences, the two Intermediate honeycombs are simply alternative cases of the same honeycomb, depending on which PP is associated with which RCL, so are in a sense mirror reflections. In the Secondary sequences, the two Intermediate honeycombs are distinct forms, but one sequence is simply the alternative case of the other, again depending on which PP is associated with which RCL; one Secondary sequence is thus the mirror reflection of the other.

2.1 Simplified Description of the Expansion Sequences.

We now consider the specific expansion sequences of the regular honeycombs. These are the 2-step sequences of the {2,3,4|2,3,4} honeycombs, in which 3D neural polyhedra might arise; and the 1-step sequences of the {2,3,3|2,3,4} honeycombs, in which 3D NEs do not arise.

Type 1 and 2 expansion sequences are easiest understood by examining the case of the $CO|OH \rightarrow CO|SR \,/\, TC|OH \rightarrow TC|SR$. Limited space only permits simple coverage here.

In a typical Type 1 {2,3,4|2,3,4} expansion sequence, we commence at the most contracted form of two sets of polyhedra, here CO and OH. There are two paths the expansion can take to reach the same full expansion. In either pathway, in the first step, one set of PPs, which are initially in contact, separate from one another, so that adjoining PPs reach a limit of unit distance from one another. They retain their relative orientations to one another, and remain coaxial on the XYZ axes; the lattice has expanded by 1 unit distance without the PPs

changing size (in this paper, lattice expansion with constant polyhedral edge length is assumed). As this lattice expands, the other set of PPs morphs into a different set of PPs. So in this particular case, as the OHs of the $[CO|OH]$ honeycomb separate from one another in a regular decentralized expansion, the other primary COs morph into TCs, to achieve the $[TC|OH]$ honeycomb. In the second step, the PP that morphed in the first step is retained, but goes through a similar expansion; while the PP that remained in the first step expansion now morphs into a different PP. So the primary TCs now expand from one another by unit distance, creating neutral OPs between them, while the other primary OHs simultaneously morph into SRs, creating neutral SPs between them.

But this is just one of the two paths that the two-step expansion from $[CO|OH]$ to $[TC|SR]$ can take. In the other path, in the first step, the other primary set of COs separate from one another in a regular decentralized expansion, creating neutral RPs between them, while the remaining primary set of OHs morph into primary SRs, to achieve the $[CO|SR]$ honeycomb. In the second step, the PPs that morphed in the first step are retained, but go through a similar unit expansion; while the PPs that remained in the first-step expansion morph into different PPs. So here, primary SRs now expand from one another by unit difference, creating neutral CBs between them, while primary COs morph into primary TCs, creating neutral OPs between them, to end in the $[TC|SR]$ honeycomb. The morphs are $OH \rightarrow SR$ and $CO \rightarrow TC$.

The other Type 1 {2,3,4|2,3,4} expansion sequences are firstly what might be regarded as the enantiomorph of that already described, which in practical terms is identical; and secondly the $VT|VT \rightarrow VT|CB \,/\, CB|VT \rightarrow CB|CB$ and $TO|TO \rightarrow TO|GR \,/\, GR|TO \rightarrow GR|GR$ sequences, which are somewhat easier to follow, as both start from PP pairs that are the same as each other, and both finish in PP pairs that are the same as each other. In addition, the two intermediate honeycombs for each of these sequences may be regarded as enantiomorphs of one other, so only one path of the two for each sequence needs to be considered.

In the primary $VT|VT \rightarrow VT|CB \,/\, CB|VT \rightarrow CB|CB$ sequence, both sets of PPs of the most contracted $[VT|VT]$ honeycomb are vertices, and the honeycomb is of zero size; all vertices coincide in the one 0D point (but with unit length virtual edges). In the first step of the expansion, one primary set of VTs expands, so that (adjacent) VTs assume unit distance from one another. The other primary set of VTs morph into primary CBs, forming a cubic lattice, whose vertices coincide with the cubic lattice of vertices of the expanded set of VTs.

In the second step of the expansion, the previously morphed primary CBs expand to unit distance from each other, and in the process adjoining pairs of primary CBs create neutral

SPs between each other. The other expanded primary VTs simultaneously morph into primary CBs, and in the process, adjoining pairs of primary CBs also form neutral SPs between each other. Each array of primary CBs and its neutral SPs forms a counterform to the other array - the two arrays are identical, but are simply displaced relative to the other.

In the greatest expansion sequence of $TO|TO \rightarrow TO|GR\ /\ GR|TO \rightarrow GR\ GR$, the Contracted form is of two sets of TOs in the $TO|TO$ honeycomb. In the first step, one set of TOs in RS face-to-face contact expands to assume unit distance, so that adjoining pairs of TOs create neutral RPs between each other, while the set of TOs morphs to become GRs, so that adjoining pairs of GRs create neutral axial OGs between each other, the pairs of GRs being in face-to-face contact. The two primary sets of TOs and GRs form the Intermediate $TO|GR$ honeycomb. In the second step, the previously morphed primary GRs in neutral OG face-to-face contact expand to unit distance apart, while the neutral axial OGs stretch to form neutral OPs. The other primary set of TOs in neutral RS face-to-face contact morphs into primary GRs, so that pairs of adjoining primary GRs develop neutral OPs between each other, the neutral RSs thus stretching into neutral OPs separating this set of GRs. Both sets of GRs and their respective sets of OPs together form the expanded $[GR|GR]$ honeycomb.

Conclusion

These honeycomb sequences offer potential for real-world structures and behavior, e.g. nanoscale engineering, environmental response, kinetic architectural and engineering structures, and dynamic structures in microgravity for deployment in Space. I hope to address this potential in future research, after first more fully describing the sequences. Proper understanding of the spatial order, including the transformations of the expansion sequences, and the remarkable consistency of the patterns that maintain unit edge length and constrained $\sqrt{1}$ and $\sqrt{2}$ orientations to the cardinal XYZ cubic lattice will enable designers to better utilize these beautiful forms, whilst encouraging further research into their coherent integrity of harmonic pattern, which reveals in some measure the extraordinary nature of empirical space.

References

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