Towards a Meta-Order of the All-Space-Filling Polyhedral Honeycombs through the Mating of Primary Polyhedra

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Abstract

My earlier polyhedral order suggests a meta-order to the all-space-filling periodical honeycombs. Polyhedral pairs mate along axes of reference cubic and tetrahedral lattices: pairs of Great Enablers *(GEs),* the positive and negative tetrahedra and truncated tetrahedra; pairs of *GEs* and the Primary Polytopes *(PPs);* and pairs of *PPs. GE:GE, GE:PP* and *PP:PP* matings correlate with the symmetry groups of the honeycombs. Matings occur in natural pairs along each axis: a *PP* mates with just two *PPs,* though one might be identical. Pairs display one-to-one correspondence with the honeycombs. Differentiating the *PPs* into two groups of four indicates an adequate meta-order.

Keywords: all-space-filling; polyhedra; honeycomb; tessellation; spatial harmony; form; order

1. Introduction

This research continues previous research into the regular and semiregular polyhedra [1]. In that work, contemplation of the fundamental spatial order of these polyhedra suggested the existence of a comprehensive pattern that would properly accommodate these entities into a satisfactory order. This order consists of three classes of polyhedra, according to their symmetry group {2,3,3}, {2,3,4}, or {2,3,5}, and extends to {2,3,6} and {2,4,4} symmetry groups to accommodate the regular and semi-regular two-dimensional tilings of the plane.

The apprehension of this elegant order of polyhedra led me to suspect the existence of a comparable order that would embrace the periodic honeycombs – the very limited number of periodic arrays of regular and semi-regular polyhedra that fill space, of which the cubic lattice is the most obvious. By "comparable" here I mean an order that would exhibit a similar elegance, beauty, and integrity; and an order that would accommodate - as particular cases - each of the all-space-filling periodic honeycombs. As with the polyhedral order, I consider it likely that alternate "colorings" of polyhedra (paralleling alternate coloring of polygonal faces in two-dimensional tilings e.g. the checker-board pattern) would extend the number of such honeycombs (paralleling the extension of the number of regular and semi-regular polyhedra from 5 and 13 to 6 and 18, respectively, with counting each pair of enantiomorphs as one, in my earlier order of the regular and semi-regular polyhedra [1]).

In the current series of 3 papers, of which this present paper is part, my intuition is that such an order ought to exist, given the intense regularity of each of its potential components. As far as I am aware, such a proper accounting of this order has yet to be presented, notwithstanding Critchlow's [2], Grünbaum's, and Shephard's valuable contributions [3, 4]. The research detailed in this series of papers consists of discerning such a meta-order.

The first paper of this series starts to address the natural order of polyhedral honeycombs [5], and should be read in conjunction with this paper. In it, I identify positive and negative tetrahedra and truncated tetrahedra as constituting what I term the four Great Enablers $GE\colon\{T^{+/-}, D^{+/-}\} = \{T^+, T^-, D^+, D^-\}$. I identify eight Primary Polytopes, in which I include the 0-D VerTex VT (hence my use here of *polytopes*, rather than *polyhedra*), together with the Truncated Octahedron, Small Rhombic cuboctahedron, Great Rhombic cuboctahedron, CubOctahedron, CuBe and Truncated Cube - $PP\colon\{VT, TO, SR, GR, OH, CO, CB, TC\}$. I also identify three restricted Neutral Elements, where they develop as 3-D polyhedra, as $NE_{rst} = \{SP, RP, OP\}$ i.e. the Square Prism, Rotated (square) Prism (rotated by π/4), and Octagonal Prism, or alternatively, seven complete secondary Neutral Elements of 0-D, 1-D, 2-D or 3-D polytopes $NE_{exp}\colon\{vt, ae, sq, og, SP, RP, OP\}$. I then present an overview of their honeycombs. Although the Square Prism and Rotated square Prism are simply cubes, there are good reasons for including these as distinct entities; and although technically the Octagonal Prism is not a regular or semi-regular polyhedron, it should also be included as a honeycomb element. These reasons become evident when viewing colored illustrations of the honeycombs, as they reveal a deeper theoretical consistency, particularly as regards expansion/contraction sequences; but as secondary elements they are of less importance.

This paper, which I regard as Part 2.5 of the series of 3, consolidates, revises, and substantially develops the research of earlier papers [6-8], in which I discern a natural order among these various polytopes, according to how they mate with one another, by meeting in regular fashion along relevant $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{3}$ cubic axes; and in which I describe how this potential order relates to the honeycombs. So I address how the GEs relate one to another; how they relate to the PPs; and how the PPs relate one to another independently of the GEs.

I structure this paper as follows: Section 2 addresses axial mating of polytopes along $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{3}$ axes of reference cubic and tetrahedral lattices. Section 3 investigates $GE\colon GE$ mating, and shows how this characterizes the singular {2,3,3|2,3,3} honeycomb. Section 4 investigates $GE\colon PP$ mating, and how this correlates with the four distinct {2,3,3|2,3,4} honeycombs. Section 5 explores $PP\colon PP$ mating, and how this correlates with the ten distinct {2,3,4|2,3,4} honeycombs; and shows how the PPs can be formally differentiated into two groups of four, this differentiation being crtitical. Section 6 presents a new structural model of interrelationship of the GEs and two groups of PPs in correlation with the honeycombs. I conclude by suggesting further research into the meta-order of the honeycombs, which envisages a shift of emphasis from mating of polyhedra to alternation of nodal lattices.

2. The Possible Axial Relationships of Polytope Pairs

The periodic honeycombs exhibit obvious reference tetrahedral or cubic lattices (depending on the specific lattice). It therefore makes sense to situate individual polyhedra (or more generally, polytopes) within an orthogonal reference system, which can accommodate both (as in Fig. 1, right). These polytopes exhibit relationships of one to another along the XYZ axes, diagonal axes, and long diagonal axes of the cube and cubic lattice; for convenience I refer to these axes as their $\sqrt{1}, \sqrt{2},$ and $\sqrt{3}$ axes, respectively. I determine the relationship of polytope to polytope on the basis of whether they are compatible or not: i.e., whether they can mate together, at either a vertex, a transverse (or occasionally axial) edge, or an axial (or occasionally transverse) face. I then differentiate this mating as proximal, where they make actual contact (vertex, edge or face); or distal, i.e. through a secondary neutral intermediary element (which might be an axial edge, neutral face, or neutral polyhedron (a prism)). For example, it is obvious that CB and VT do not mate along a $\sqrt{1}$ axis (square-to-vertex), nor do they mate along a $\sqrt{2}$ axis (edge-to-vertex), but they do mate along a $\sqrt{3}$ axis (vertex-to-vertex). Again, TO and OH do not mate along a $\sqrt{1}$ axis (rotated square-to-vertex), nor do they mate along a $\sqrt{3}$ axis (hexagon-to-triangle), but they do mate along a $\sqrt{2}$ axis (diagonal transverse edge-to-edge).

3. How do GE: GE Pairs mate?

I first consider the GEs, which differ from the PPs and NEs in that they do not develop {2,3,4} symmetry, and develop only {2,3,3} symmetry on just the $\sqrt{1}$ axes and on alternating α and β tetrahedral $\sqrt{3}$ axes. Without loss of generality, I define positive and negative Ts to be as shown in Fig. 1, left; and the positive and negative Ds to be those that are then developed from their respective solid.

These matrices clearly show that GE: GE axial matings always occur in pairs, e.g. on the $\sqrt{1}$ axis, T^+ mates with T^- and D^- ; on the $\sqrt{3}_{\beta}^{\alpha}$ axes, T^+ mates with D^+ and T^-. This axial mating of a particular polytope with a specific pair of polytopes applies in general.

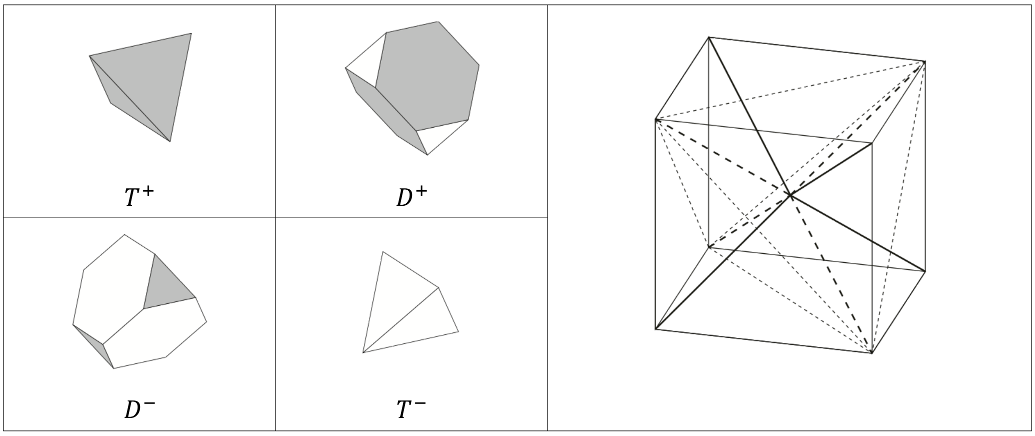


Fig. 1. Left: The four Great Enablers *(GEs)*: D^+ is the truncation of T^+, D^- of T^-.

Right: $\boldsymbol{\alpha}$ (dashed) and $\boldsymbol{\beta}$ (solid) √3 axes within T^+, within its cube.

Table 1. Matrices of *GE:GE* pairs from top left √1, top right $\boldsymbol{\alpha}$√3,

bottom left $\boldsymbol{\beta}$√3, and bottom right both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ √3 axes.

	down, from above	╲	╲	╱	╱
up, from below	$\sqrt{1}$	T^+	D^+	T^-	D^-
╱	T^+			╱	╱
╱	D^+			╱	╱
╲	T^-	╲	╲		
╲	D^-	╲	╲		

α	down, from above	△	⬡	⫯	▽
up, from below	$\sqrt{3}$	T^+	D^+	T^-	D^-
⫰	T^+			⊙	
△	D^+	△			
▽	T^-				▽
⬡	D^-		⬡		

β	down, from above	⫯	▽	△	⬡
up, from below	$\sqrt{3}$	T^+	D^+	T^-	D^-
▽	T^+		▽		
⬡	D^+				⬡
⫰	T^-	⊙			
△	D^-			△	

$\sqrt{3}$	T^+	D^+	T^-	D^-
T^+		▽	⊙	
D^+	△			⬡
T^-	⊙			▽
D^-		⬡	△	

For the GEs, firstly, the $\sqrt{2}$ diagonal edge elements of a polyhedron on each $\sqrt{1}$ axis alternate in orientation from top/side to bottom/opposite side. Secondly, the facial elements on each pair of coaxial α and β $\sqrt{3}$ axes change between triangles and hexagons, both of which have an associated orientation. This orientation is obvious in the case of the triangles, which demonstrate a 180° rotational phase shift (which I show as up- or down-ward pointing, though in a honeycomb these lie in multiple directions). In the case of the hexagons, I indicate this in notation by the appendage of an extended triangle; for vertices, a small line.

Figure 2 shows the axial mating patterns. The matrices reveal the quite highly constrained proper relations between GE and GE. A GE mates with the same polyhedron of opposite sign, or with the other GE polyhedron of opposite sign along the $\sqrt{1}$ axes; but it does not properly mate with either GE of its own sign along those axes (it doesn't mate with itself, or with the other polyhedron of the same sign). A GE only mates with one of its opposite sign, or with the other GE of the same sign as itself along the α and β $\sqrt{3}$ axes; but it does not properly mate with itself, or with the other GE polyhedron of the opposite sign along those axes.

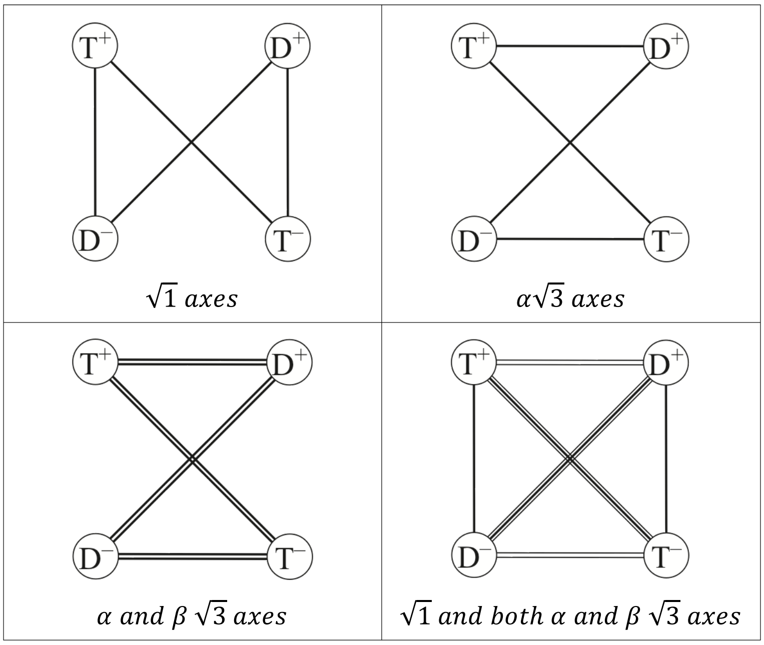


Fig. 2. GE pairings: on √1; $\boldsymbol{\alpha}$ √3; $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ √3; and √1 and both √3 axes.

The singular $\{2,3,3|2,3,3\}$ honeycomb (of tetrahedra and truncated tetrahedra) meets these constraints. In my earlier paper on the polyhedral honeycombs [7], I describe this particular honeycomb as a four-way alternation, or mix-and-match. This solitary honeycomb provides four permutations, according to which GE associates with which reference tetrahedral lattice, i.e.:

$$\begin{Vmatrix} D^- & T^- \\ D^+ & T^+ \end{Vmatrix}, \quad \begin{Vmatrix} D^+ & D^- \\ T^+ & T^- \end{Vmatrix},$$

$$\begin{Vmatrix} T^+ & D^+ \\ T^- & D^- \end{Vmatrix}, \quad \begin{Vmatrix} T^- & T^+ \\ D^- & D^+ \end{Vmatrix}.$$

4. How do *GE:PP* pairs mate?

Consider now the potential relations between GEs and PPs. On the $\sqrt{1}$ axes, GEs develop only transverse diagonal edges, but no PP does (for OH and TO, the transverse edge is on the $\sqrt{2}$ axis, and the $\sqrt{1}$ axis element of the TO is the rotated square, which is merely bounded by non-axial diagonal edges). Hence, they do not mate on these $\sqrt{1}$ axes. The GEs do not develop symmetry on $\sqrt{2}$ axes, so they do not mate on those axes. Therefore, how do GEs and PPs relate on the $\sqrt{3}$ axes?

Once again, we observe that the potential matings of polytopes are constrained. On the α and β $\sqrt{3}$ axes, Ts mate only with CBs or VTs, by vertex; or with SRs or OHs, by downward pointing triangle. Ds mate only with TCs and COs, by upward pointing triangle; or with GRs and TOs, by hexagon. Each GE mates with one set of four of the PPs:

$$T: \{\ CB, VT; SR, OH\ \}$$

$$D: \{\ GR, TO; TC, CO\ \}$$

Tables 2 and 3 show the pairings, and arrays. Note the arrowed expansion sequences.

Table 2. Matrix of *GE:PP* for $\boldsymbol{\alpha} + \boldsymbol{\beta}$ √3 Axes.

$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
T			⊙	▽	← – –	– – –	⊙	▽
D	⬡	△	← – –	– – –	⬡	△		

Table 3. Corresponding matrix of {2,3,3 | 2,3,4} Arrays.

$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
T			$\begin{vmatrix} T^+ & CB \\ SR & T^- \end{vmatrix}$		← – – – –		$\begin{vmatrix} T^+ & VT \\ OH & T^- \end{vmatrix}$	
D	$\begin{vmatrix} D^+ & TC \\ GR & D^- \end{vmatrix}$		← – – – –		$\begin{vmatrix} D^+ & CO \\ TO & D^- \end{vmatrix}$			

Taken together, Tables 2 and 3 show that the pairings of mateable polyhedra of Table 2 for the $\sqrt{3}$ PP axes correlate with the possible {2,3,3|2,3,4} arrays of Table 3, i.e.:

$$\begin{vmatrix} T^+ & VT \\ OH & T^- \end{vmatrix}, \quad \begin{vmatrix} T^+ & CB \\ SR & T^- \end{vmatrix}, \quad \begin{vmatrix} D^+ & CO \\ TO & D^- \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} D^+ & TC \\ GR & D^- \end{vmatrix},$$

together with their permutations.

5. How do *PP:PP* pairs mate?

Using a similar procedure, I compare pairs of PP along the $\sqrt{1}, \sqrt{2}, \sqrt{3}$ axes, to determine their relationships of one to another on the basis of whether they are compatible or not, i.e. whether they mate, at either a vertex, a (transverse or occasionally axial) edge, or an (axial or occasionally transverse) face. In the case of $PP\colon PP$, for $\{2,3,4|2,3,4\}$ symmetry, this can be proximal, where they make actual contact (by vertex, edge or face) - or distal, through a secondary neutral intermediary (which could be an axial edge, neutral polygonal face, or neutral polyhedron - a prism). While this could also be explored for the $GE\colon GE$ and the $GE\colon PP$ cases, it is not applicable to the *all-space-filling* periodic honeycombs, which are the subject of this research. Nor is the potential use of *antiprismatic neutral elements,* particularly the OH, to deal with the alternation in orientation of triangular faces, though rather intriguing arrays can be imagined from the $[CO|OH]$ that are not all-space-filling e.g. $[TC|OH]$, and $[SR|OH]$.

This comparison is striking, and demonstrates the same behavior observed earlier for the $GE\colon GE$ and $GE\colon PP$ matings. *For each axial case, the matings of polytopes form natural pairs, and these pairs differ for each axis.* This formal behavior is beautiful to appreciate.

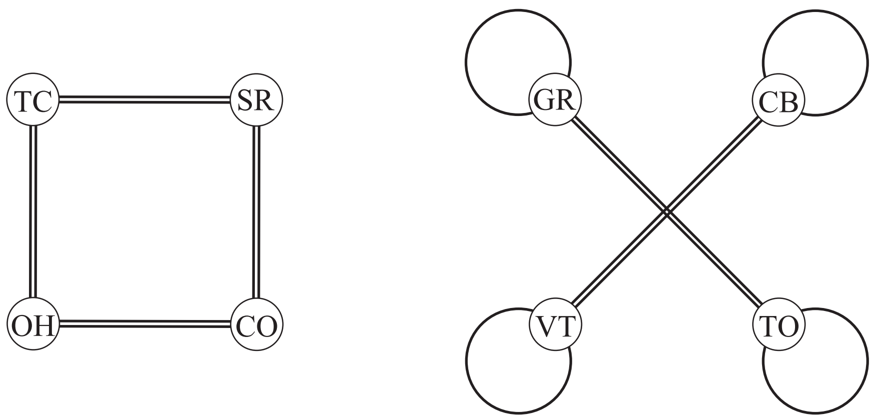


Fig. 3. Two groups of four *PPs* with √3 axis matings: left: non-self-reflexive *PPs* share double relations with each of two neighbors (e.g. *OH:CO, CO:OH, CO:SR, SR:CO*), and do not mate with their opposite; and right: self-reflexive *PPs* where larger circles denote self-reflexive partnerships (e.g. *GR:GR),* and diagonals denote double relations with opposites (e.g. *GR:TO, TO:GR*), but *PPs* do not mate with their neighbors.

Figure 3 shows that in this $PP\colon PP$ case, further complexity develops on the $\sqrt{3}$ axes. (In this figure, the connections are directed; so in the right figure of self-reflexive PPs, though it might appear there are four lines of connection (2 straight diagonals, 2 ends of the self-reflexive circle), two (one shown as straight, one circular) are outward ("mater") and two are inward ("matee"), as each PP mates with just two PPs: itself and its diagonal opposite).

Figure 3 above and Table 4 below show that these natural pairs for the various axes are:

√1 pairs: $\sqrt{1}\ \text{pairs} \quad : \quad \{\ (\text{GR}, \text{TC}), (\text{CB}, \text{SR}), (\text{TO}, \text{CO}), (\text{VT}, \text{OH})\ \}$

As noted above, Table 3 shows these √1 pairs accord with the {2,3,3|2,3,4} honeycombs:

$$\begin{vmatrix} D^+ & TC \\ GR & D^- \end{vmatrix}, \quad \begin{vmatrix} T^+ & CB \\ SR & T^- \end{vmatrix}, \quad \begin{vmatrix} D^+ & CO \\ TO & D^- \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} T^+ & VT \\ OH & T^- \end{vmatrix}.$$

Taken together with their self-reflexive pairings:

$$= \{\ \text{GR}: (\text{GR}, \text{TC}), \text{CB}: (\text{CB}, \text{SR}), \text{TO}: (\text{TO}, \text{CO}), \text{VT}: (\text{VT}, \text{OH}),$$
$$\text{TC}: (\text{TC}, \text{GR}), \text{SR}: (\text{SR}, \text{CB}), \text{CO}: (\text{CO}, \text{TO}), \text{OH}: (\text{OH}, \text{VT})\ \}$$

√2 pairs: $\sqrt{2}\ \text{pairs} \quad : \quad \{\ (\text{GR}, \text{SR}), (\text{TC}, \text{CB}), (\text{TO}, \text{OH}), (\text{CO}, \text{VT})\ \}$

These pairs do not correlate with any honeycomb. Together with self-reflexive pairings:

$$= \{\ \text{GR}: (\text{GR}, \text{SR}), \text{CB}: (\text{CB}, \text{TC}), \text{TO}: (\text{TO}, \text{OH}), \text{VT}: (\text{VT}, \text{OH}),$$
$$\text{TC}: (\text{TC}, \text{CB}), \text{SR}: (\text{SR}, \text{GR}), \text{CO}: (\text{CO}, \text{VT}), \text{OH}: (\text{OH}, \text{TO})\ \}$$

√3 pairs: $\sqrt{3}\ \text{self-reflexive pairs} \quad : \quad \{\ (\text{GR}, \text{TO}), (\text{CB}, \text{VT})\ \}$

$$= \quad \{\ \text{GR}: \text{GR}, \text{TO}: \text{TO}, \text{CB}: \text{CB}, \text{VT}: \text{VT}, \text{GR}: \text{TO}, \text{TO}: \text{GR}, \text{CB}: \text{VT}, \text{VT}: \text{CB}\ \}$$

$$\sqrt{3}\ \text{non-self-reflexive pairs} \quad : \quad \{\ \text{OH}: \text{CO}, \text{OH}: \text{TC}, \text{SR}: \text{CO}, \text{SR}: \text{TC}\ \}$$

$$= \quad \{\ \text{CO}: (\text{OH}, \text{SR}), \text{OH}: (\text{TC}, \text{CO}), \text{TC}: (\text{SR}, \text{OH}), \text{SR}: (\text{CO}, \text{TC})\ \}$$

Table 6 shows √3 pairs correlate one-to-one with corresponding {2,3,4|2,3,4} honeycombs:

$$[GR|GR], [TO|TO], [CB|CB], [VT|VT], [GR|TO], [TO|GR], [CB|VT], [VT|CB],$$
$$[CO|OH], [CO|SR], [OH|TC], [OH|CO], [TC|SR], [TC|OH], [SR|CO], [SR|TC].$$

On the $\sqrt{1}$ and $\sqrt{2}$ axes, each PP is self-reflexive - it mates with itself, as well as with just one other, its pair. However, on the $\sqrt{3}$ axes, the situation changes. While the four PPs that do not have triangular faces are self-reflexive - each mates with itself, while it also mates with just one other, the other four PPs that do have triangular faces, which may alternate in rotational phase angle (i.e. orientation - point up or down). This means that those PPs are non-self-reflexive, as each PP cannot mate with itself, because the direction of apex flips between upper and lower. Placing these four in square array, Fig. 3 shows that each mates with its two neighbors, but not with its opposite. In $\sqrt{3}$ matrices, common mating conditions, situated in overlapping squares, accord with the expansion/contraction sequences of arrays discussed in my earlier paper in this series [5] (which I recommend be read with this paper).

The patterns of pairings on the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ √3 axes of $GE{:}\,GE$, $GE{:}\,PP$ and $PP{:}\,PP$ can be developed into the two-dimensional order shown in Fig. 4 below.

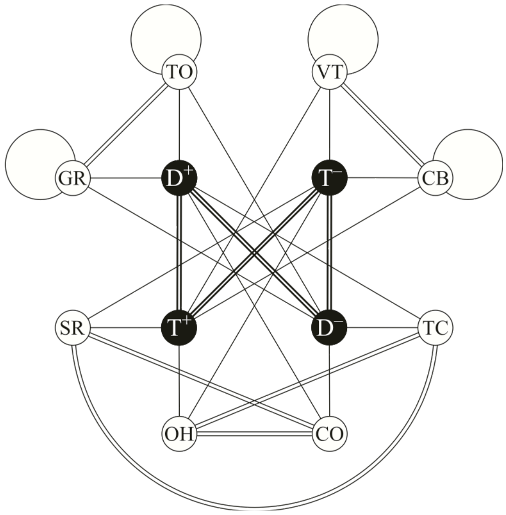


Fig. 4. *GE:GE, GE:PP* and *PP:PP* pairings on both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ √3 axes.

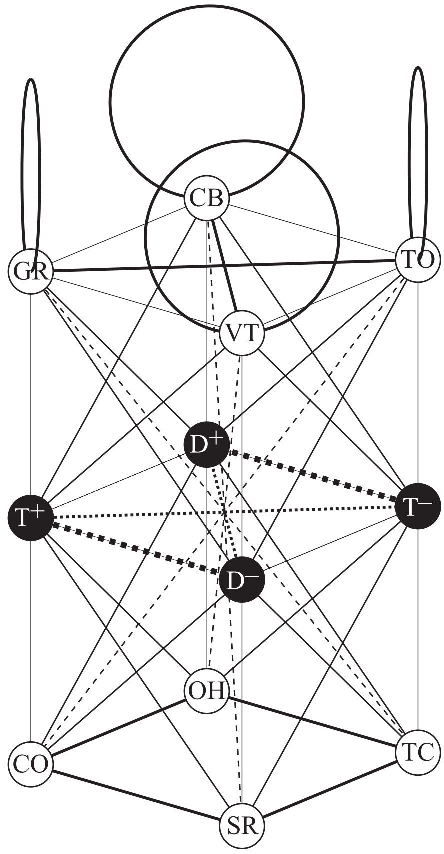


Fig. 5. Three-dimensional arrangement of the four GEs and eight PPs together with their linkages that reveals the meta-order of the all-space-filling periodic honeycombs.

Notes to Fig. 4: The four GEs form an inner square arrangement, while the eight PPs form an outer octagonal arrangement. Lines, circular and semi-circular arcs represent directed matings. $GE{:}GE$ matings are shown in heavier line width. The circular arcs at top attached to GR, TO, VT and CB represent self-reflexive matings, their ends depicting both "mater" and "matee". To aid clarity, the two lines connecting SR and TC are shown at bottom as semicircular arcs. The two groups of PPs are clearly distinct, the self-reflexive group above, the non-self-reflexive below. Each PP has a unique opposite PP, e.g. $GR{:}TC$.

GE:GE matings: As described earlier, each GE mates (and is mated with) its polyhedron of opposite orientation (sign), and the other polyhedron of the same orientation (sign), but on these α and β √3 axes does not mate with the other polyhedron of opposite orientation (sign).

PP:PP matings: In the non-self-reflexive group of four below, each PP mates and is mated with two other PPs, but not with the fourth. In the self-reflexive group of four above, each PP mates and is mated with one other PP of that group, and with itself.

GE:PP matings: Each PP mates and is mated with two GEs that are the same polyhedra but of opposite orientations (i.e. positive and negative signs). Within each group of four PPs, two PPs mate or are mated with one GE, and are mated or mate with the other GE that is the same polyhedron, but of different orientation (sign). The other two PPs in that group of four mate or are mated with the GE that is the other polyhedron and of opposite orientation (sign), and are mated or mate with that same other polyhedron of the 'opposite to the opposite', i.e. the same orientation (sign). Each GE mates or is mated with one pair of PPs from one group of 4, and is mated or mates with the opposite pair on the octagon from the other group of four.

Table 4. Matrices of *PP:PP* for top left: √1, top right: √2, and bottom left: √3 Axes.
Bottom right: Axes that facilitate mating for the *PPs*,
where √1,2,3 represents √1, √2, √3; and √1,2 represents √1, √2.

$\sqrt{1}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	⯃	⯃						
TC	⯃	⯃						
CB			□	□				
SR			□	□				
TO					◇	◇		
CO					◇	◇		
VT							⊙	⊙
OH							⊙	⊙

$\sqrt{2}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	□			□				
TC		┼	┼					
CB		┼	┼					
SR	□			□				
TO					×			×
CO						⊙	⊙	
VT						⊙	⊙	
OH					×			×

$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	⬡				⬡			
TC				✡				✡
CB			⊙				⊙	
SR		✡				✡		
TO	⬡				⬡			
CO				✡				✡
VT			⊙				⊙	
OH		✡				✡		

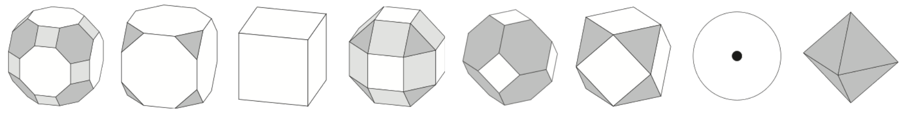
axes	GR	TC	CB	SR	TO	CO	VT	OH
GR	$\sqrt{1,2,3}$	$\sqrt{1}$		$\sqrt{2}$	$\sqrt{3}$			
TC	$\sqrt{1}$	$\sqrt{1,2}$	$\sqrt{2}$	$\sqrt{3}$				$\sqrt{3}$
CB		$\sqrt{2}$	$\sqrt{1,2,3}$	$\sqrt{1}$			$\sqrt{3}$	
SR	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{1}$	$\sqrt{1,2}$		$\sqrt{3}$		
TO	$\sqrt{3}$				$\sqrt{1,2,3}$	$\sqrt{1}$		$\sqrt{2}$
CO				$\sqrt{3}$	$\sqrt{1}$	$\sqrt{1,2}$	$\sqrt{2}$	$\sqrt{3}$
VT			$\sqrt{3}$			$\sqrt{2}$	$\sqrt{1,2,3}$	$\sqrt{1}$
OH		$\sqrt{3}$			$\sqrt{2}$	$\sqrt{3}$	$\sqrt{1}$	$\sqrt{1,2}$

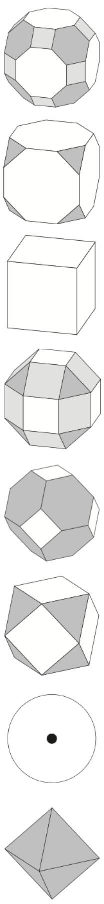
Table 5. Matrices of √1, √2, √3 axes that facilitate mating for the two groups of *PPs*.

axes	TC	SR	CO	OH
TC	$\sqrt{1}, \sqrt{2}$	$\sqrt{3}$		$\sqrt{3}$
SR	$\sqrt{3}$	$\sqrt{1}, \sqrt{2}$	$\sqrt{3}$	
CO		$\sqrt{3}$	$\sqrt{1}, \sqrt{2}$	$\sqrt{3}$
OH	$\sqrt{3}$		$\sqrt{3}$	$\sqrt{1}, \sqrt{2}$

axes	GR	CB	TO	VT
GR	$\sqrt{1}, \sqrt{2}, \sqrt{3}$		$\sqrt{3}$	
CB		$\sqrt{1}, \sqrt{2}, \sqrt{3}$		$\sqrt{3}$
TO	$\sqrt{3}$		$\sqrt{1}, \sqrt{2}, \sqrt{3}$	
VT		$\sqrt{3}$		$\sqrt{1}, \sqrt{2}, \sqrt{3}$

Table 6. Matrix of {2,3,4 | 2,3,4} honeycombs with overlapping squares of expansion sequences.





$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	$[GR \mid GR]$				$[GR \mid TO]$			
TC				$[TC \mid SR]$				$[TC \mid OH]$
CB			$[CB \mid CB]$				$[CB \mid VT]$	
SR		$[SR \mid TC]$				$[SR \mid CO]$		
TO	$[TO \mid GR]$				$[TO \mid TO]$			
CO				$[CO \mid SR]$				$[CO \mid OH]$
VT			$[VT \mid CB]$				$[VT \mid VT]$	
OH		$[OH \mid TC]$				$[OH \mid CO]$		

6. Development of the Order of All-space-filling Periodic Honeycombs

I develop Figure 4 into the three-dimensional arrangement of GEs and PPs of Fig. 5 shown at right that reveals the coherent integral meta-order of the various periodic honeycombs. In this arrangement, secondary neutral elements are omitted for the sake of clarity. The structure consists of a vertically disposed double cube, combined with various links between the nodes. The upper stratum consists of the group of four self-reflexive PPs. The middle stratum consists of the GEs. The lower stratum consists of the group of four non-self-reflexive PPs. Polar opposites of PPs across the dashed long diagonals of the double cube uniquely relate PPs from one group to the other, so each PP associates with its specific opposite, e.g. $GR - TC$. (Note that axial linkages do not necessarily accord with the axial relationships of polyhedra within any particular honeycomb; for example, a √2 link in Fig. 5 might represent an actual √1 or √3 axial relationship in the corresponding honeycomb).

Circular links through each of the four GEs of the upper stratum represent the four **self-reflexive {2,3,4}|{2,3,4}** honeycombs: $[GR|GR]$, $[TO|TO]$, $[CB|CB]$, and $[VT|VT]$. For clarity, I've folded these links into the vertical, although they belong in the plane of the upper square. √2 diagonal links of the horizontal square of the upper stratum represent **the other {2,3,4}|{2,3,4}** honeycombs of their group: $[GR|TO]$, $[TO|GR]$, $[CB|VT]$, and $[VT|CB]$.

√1 edge links of the horizontal square of the lower stratum represent the eight **non-self-reflexive {2,3,4}|{2,3,4}** honeycombs of the other group of four PPs: $[SR|TC]$, $[SR|CO]$, $[OH|TC]$, and $[OH|CO]$; and $[TC|SR]$, $[TC|OH]$, $[CO|SR]$, and $[CO|OH]$. (The expansion/ contraction sequences of these 16 honeycombs of upper circular, upper √2, and lower √1 links are clearly displayed in Fig. 1 of my Ref. [5], which is Part I of this series of papers; that figure also details the specific neutral elements for each array).

The **{2,3,3}|{2,3,4}** honeycombs are represented by polar opposites of PPs that are mediated by their respective GEs, so that each of these honeycombs is represented as an inclined diamond: $\begin{vmatrix} D^- & GR \\ TC & D^+ \end{vmatrix}$, $\begin{vmatrix} D^- & TO \\ CO & D^+ \end{vmatrix}$, $\begin{vmatrix} T^+ & CB \\ SR & T^- \end{vmatrix}$, and $\begin{vmatrix} T^+ & VT \\ OH & T^- \end{vmatrix}$. (Each diamond and {2,3,3}|{2,3,4} honeycomb then has four permutations, depending upon which of the four constituent polytopes is located at which node of the reference lattices).

The singular **{2,3,3}|{2,3,3}** honeycomb is represented by the dashed pair of √1 edge and dashed pair of √2 diagonal links of the horizontal square of the middle stratum at the center of the model. Thicker dashed lines represent √1 axial relationships; dashed √2 diagonals represent √3 axial relationships. (This honeycomb also comes in 4 permutations, according to which of the four constituent GEs is located at which node of the reference lattices).

I thus advance a coherent model of the meta-order of the all-space-filling periodic honeycombs. This has the great advantage of imageability: this structural model can be readily memorized, then projected into the space of the imagination, and employed to deduce or recall the various honeycombs that emerge as specific cases of its integral coherence.

However, in fairness, I feel the model and its inherent order, whilst offering pragmatic and pedagogic advantage, lacks a certain elegance that I anticipated that it should reveal. Firstly, the circular self-reflexive links detract from the purity of the model, and whilst I suspect a four-dimensional structure might be better employed, possibly through hypercubes, to date I have been unable to derive a more satisfactory form. Secondly, the status of the {2,3,3}|{2,3,3} honeycomb still troubles me - it should be core, but I'm as yet unclear as to how it meaningfully relates to the other honeycombs, beyond being an incidental recombination of elements of the base octet truss $\begin{vmatrix} T^+ & VT \\ OH & T^- \end{vmatrix}$, which might instead be core.

I therefore intend to refine the existing model in further research, paying particular attention to the alternation of primary nodes and lattices, and to present that order in the final Part 3 of this series. I anticipate a shift in emphasis from *"what mates with what"*, to *"what alternates with what"*. This will likely emphasize how the various honeycombs may be understood as different vibrational states of the same rare ethereal structure of space.

7. Conclusion

Inspired by my recognition of an adequate order to describe the regular and semi-regular polyhedra, I continue the research in this paper into a comprehensive order to properly account for the all-space-filling polyhedral honeycombs, by investigating how pairs of polyhedra combine. Having identified four Great Enablers and eight Primary Polytopes, I consider how $GE{:}\,GE$, $GE{:}\,PP$, and $PP{:}\,PP$ pairs combine with one another, proximally or distally, along their $\sqrt{1}$, $\sqrt{2}$, or $\sqrt{3}$ axes, and how these diverse matings relate to specific honeycombs. In this process, I regard certain polytopes as secondary neutral elements that lie between GEs, or between GEs and PPs, or between PPs (dependent upon the symmetry case). Except for the core {2,3,3}|{2,3,3} honeycomb, which I regard as the most convoluted, energetic form, honeycombs are parts of expansion/contraction sequences, these being two-stage for the {2,3,3}|{2,3,4} honeycombs, and three-stage for the {2,3,4}|{2,3,4} honeycombs. I detail the various honeycombs and their expansion sequences in Ref. [5].

Here, I show that the matings are highly constrained. Matings always occur in pairs. In the possible honeycombs, on a particular axis, a given polytope mates with just one polytope, and separately, with just one other polytope. For the given polytope, these pairs of mateable polytopes vary by axis. These pairs also vary by symmetry group – for any one symmetry group and axis, a constituent polytope pairs with just two others, and that association pattern is unique to the symmetry group and axis. In the case of the $PP{:}\,PP$ matings of the $\{2,3,4|2,3,4\}$ symmetries, in general one of these matings is with itself, the exceptions being $\sqrt{3}$ axis matings. The characteristics of these $\sqrt{3}$ axis matings enable me to formally differentiate the PPs into two groups of four, which I show as two squares. $PP{:}\,PP$ pairings of the first group behave in a similar manner to $GE{:}\,GE$ and $GE{:}\,PP$ pairings, with PPs pairing with themselves and with their opposites. Conversely, those of the second group do not. Instead, each PP pairs with its two neighbors, but not with itself, or with its opposite. For the $GE{:}\,PP$ and $PP{:}\,PP$ pairings, the expansion/contraction sequences of my earlier paper [5] are evident in the $\sqrt{3}$ matrices; in particular for the $PP{:}\,PP$ pairings, the sequences are evident (in this paper) as overlapping squares in Table 4 (bottom left) and in Table 6.

I therefore move beyond the mere recognition of sets of GEs and PPs, to an appreciation of the profound inner order that relates these individual elements, according to their potential to mate with one another, and that correlates these matings with the proper honeycombs that they form (together with the neutral elements). This research effort respects prior efforts, whilst seeking to surpass them. The challenge now to evince a more adequate formal representation of the profound harmony that is here merely glimpsed, and in a future paper to more adequately describe that new meta-order of the all-space-filling periodic honeycombs.

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References

[1] R. C. Meurant, "A New Order in Space - Aspects of a Three-fold Ordering of the Fundamental Symmetries of Empirical Space, as evidenced in the Platonic and Archimedean Polyhedra - Together with a Two-fold Extension of the Order to include the Regular and Semi-regular Tilings of the Planar Surface", International Journal of Space Structures, vol. 6, no. 1, (1991), University of Surrey, Essex, pp. 11-32; also accessible as Paper 06P at http://rmeurant.com/its/harmony.html .

[2] K. Critchlow, "Order in Space". Thames and Hudson, London, (1969).

[3] B. Grünbaum, "Uniform Tilings of 3-space", Geombinatorics 4, (1994), pp. 49–56.

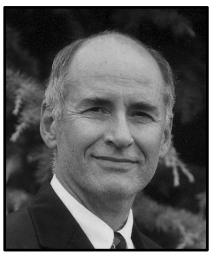
[4] B. Grünbaum and G. C. Shephard, "Tilings and Patterns". W. H. Freeman, NY, (1987).

[5] R. C. Meurant, "Towards a New Order of the Polyhedral Honeycombs: Part I: The New Order Introduced". In Kim, T.-h., CCIS/LNCS: DCA2012 Digital Contents and Applications, Springer-Verlag, Berlin Heidleberg, (2012), pp. 217–225; also accessible as Paper 57P at http://rmeurant.com/its/lict-4.html .

[6] R. C. Meurant, "A New Order of the Polyhedral Honeycombs II: Which Polyhedron Mates with Which?". Proceedings of The 4th International Conference on Digital Contents and Applications, Jeju, Korea (2015), December 16-19.

[7] R. C. Meurant, "Towards a New Order of the Polyhedral Honeycombs: Part II: Who Dances with Whom?", in L. Li and T.-W. Kuo (eds.), Proceedings of The Seventh International Conference on Information, Taipei, (2015), November 25-28, pp. 369-373.

[8] R. C. Meurant, "Mating of Polyhedra to Evince an Order of the All-Space-Filling Periodic Honeycombs", in the International Journal of u- and e- Service, Science and Technology, Vol. 9 (2016), (in press).



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