A Novel 2.5D Schema of the Regular and Semi-regular Polytopes and their Sequences, with Analysis by Polytope and Surface Elements

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1. Abstract

This paper revises my previous order of the regular and semi-regular polytopes, in the light of more recent explorations of the regular and semi-regular polyhedral honeycombs and tessellations of the plane, to advance a novel 2.5D cubic schema. First, a Verticial Polytope (VP) is added to each of the 5 classes, increasing the number of Primary Polytopes (PPs) from 7 to 8 per class. Second, the generic order of each class is restructured from a "⊥" of two long sequences, to locate the PPs at the nodes of a 3D cube; vertically disposed in pairs, they form the upper and lower square of the cubic schema, to present in 2D as 2 overlapping diamonds. This enables the structural motif of sequence clusters previously discovered in the honeycombs and tessellations to be applied to the individual polyhedra and tessellations of each class. The diamond edges form the 2 steps of the sequences of development. In the lower diamond, the lower seed VP expands in 2 ways to 2 intermediary Polar Polytopes (PLs), which expand again to meet in the common Small Rhomb (SR). This SR jitterbugs through the Snub enantiomorphs at the cubic center to form the seed Quasi-Regular (QR) of the upper diamond. The pattern repeats as the upper seed QR expands in 2 ways to the 2 Truncated Polars (TPs), which expand again in 2 ways to meet in the common Great Rhomb (GR). At each first step of each sequence, the surface polar (+ & –) polytopes of the PP either separate by unit distance, or morph. At each second step, they either morph or separate. The pattern also applies to the individual surface (+ & –) polytopes of each PP. I characterize the PPs and (+/0/–)ve surface polytopes by class, and show the rigorous correspondence between classes.

Keywords: polyhedra, tessellation, honeycomb, structural morphology, semi-regular, order.

2. Contributions of this paper

- The prior order of the individual Platonic and Archimedean polyhedra and tessellations is revised, in light of explorations of the periodic polyhedral honeycombs and tilings.
- A virtual verticial polytope of coincident +/0/–ve VTs is added to the PPs of each class.
- Within each class, the archetypical structure is changed from two intersecting sequences in inverted T form "⊥", to a 2.5D cubic (cuboid for convenience) schema.
- The now eight PPs of each class are disposed in pairs at the nodes of the upper and lower faces of the nominal cube, to form in projection an upper and lower diamond, to locate L/R snubs at the center of the cube, and of the long diagonal axis SRQR—QR.
- Within each class, expansion sequence clusters derived from the honeycomb explorations are recognized for the two (+ve & –ve) PLs and 2 sets of (+ve & –ve) prs.
- Each 2-step expansion sequence is characterized by firstly, a polarizing transformation (neutral to polar); and secondly, a neutralizing transformation (polar to neutral) (Fig. 6).

- A seed neutral verticial VP at the base of the lower sequence cluster develops in either of two ways into two intermediary PL polytopes; those two polytopes then develop in the other way, respectively, to culminate in a common neutral small rhombic polytope.
- The SR polytope then jitterbugs [5] through either left/right enantiomorph to contract into the QR polytope, thus becoming the seed polytope of the upper sequence cluster.
- That neutral QR similarly develops in either of two ways to 2 intermediary polar TPs, which then develop in the other way, to culminate in the common neutral GR polytope.
- The sequence cluster progressions are characterized by the corresponding transformations of the 5×3×4=60 (+/0/–)ve surface vertices, edges, & faces of the PPs.
- The new order is consistent across all 5 classes, 40 PPs, and 60 (+/0/–) elements.
- Information sheets are provided of the generic schema and each class of the polytopes.

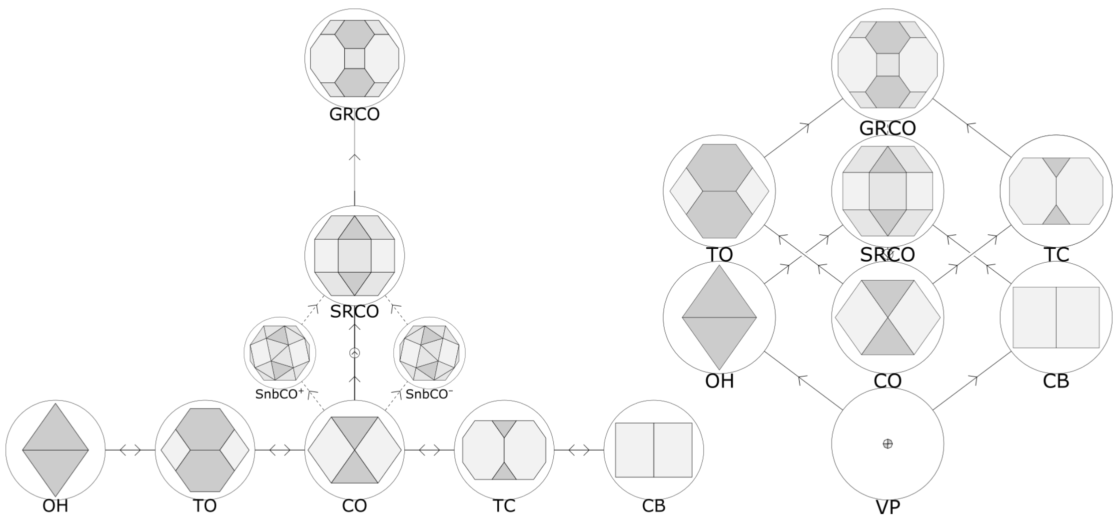


Fig. 1: The old (left) and new (right) orders of the PPs and Snub of Class II of {2,3,4} symmetry. The new order adds a key null VP, arranging the PPs as overlapping diamonds of lower & upper layer.

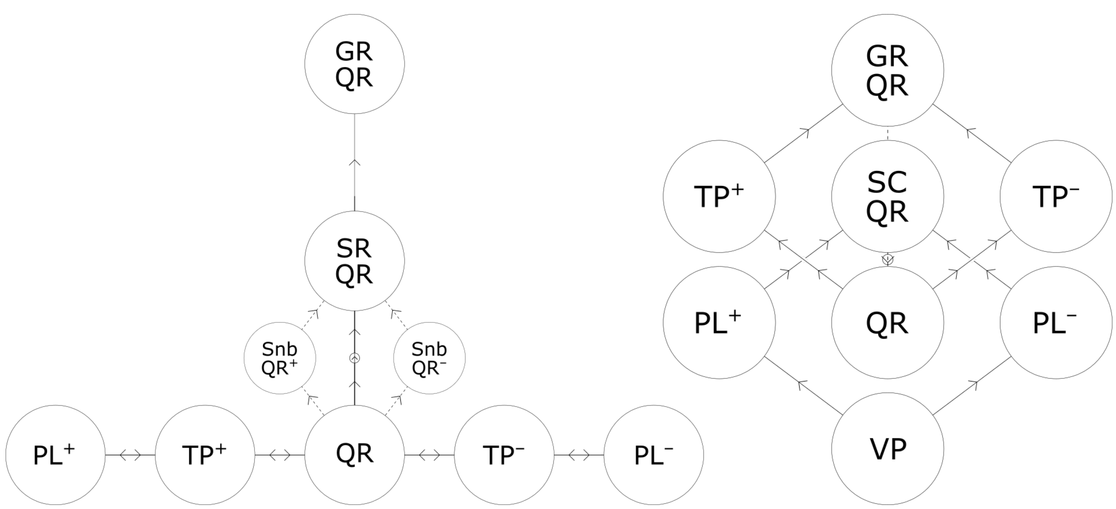


Fig. 2: The old & new generic orders of the PPs and Snub. The new order locates the PPs in vertical pairs; lower & upper diamond layers show the corresponding sequence clusters VP–PL^+/PL^-–SRQR & QR–TP^+/TP^-–GRQR mediated by the QR–SnbQR–SRQR jitterbug (hidden) [5] on long cube diagonal.

3. Introduction

Following Critchlow's Order in Space [1], I earlier advanced an order in space to account for the regular & semi-regular polyhedra in classes I–III of {2,3,3}/{2,3,4}/{2,3,5} symmetry, and the tessellations in classes IV & V of {2,3,6}/{2,4,4} symmetry [2]. Each class of 7 PP had an 8th snub transitional polytope. The PPs were presented in inverted "T" form, of a horizontal base truncation sequence between (+&–)ve polar polytopes PL^{+}–TP^{+}–QR–TP^{-}–PL^{-} and a vertical transcendence sequence QR–SnbQR–SR–GR. The PDF of that research is downloadable [2]. A relevant critical insight from a related paper was the recognition of the regular polytopes of the (+/–)ve tetrahedra, octahedron & cube, icosahedron & dodecahedron, triagonal and hexagonal arrays, and (+/–)ve square arrays as polar forms, mediated by the perfect forms of the quasiregular polytopes (QR) of the respective tetratetrahedron (2-colored OH), octahexahedron (CO), icosidodecahedron, and tri–hex and square–square arrays [3].

This paper advances a cubic 2.5D schema to better describe the structural morphology of each class, by introducing a null polytope VP. The structural order of each class is transformed from a "⊥" form of 2 sequences, into 2 clusters: in either sequence, a seed polytope either expands in one of 2 ways as its surface polytopes separate, or morphs into a different surface polytope, into 2 kinds of intermediary polytope, then in the other way, to their fully developed polytope. Paper [4] described this for Class II of {2,3,4} symmetry, which PPs correspond to those that constitute all 16 Class III honeycombs; and to confirm the order validity, described Class IV of {2,3,6} symmetry of the tessellations, which is asymmetric: polar opposites are not the same recolored polytopes, but differ. This paper describes the complete novel order in space, characterizing the polytopes and their expansion sequences and clusters by PP & Snb, and by the 5×3×4 (×2)=60 (120) class × polarity × facial type (× separation) vertex/edge/face surface PP elements. The diverse correspondences of the order across classes are rigorous. The work merits sustained contemplation of the figures.

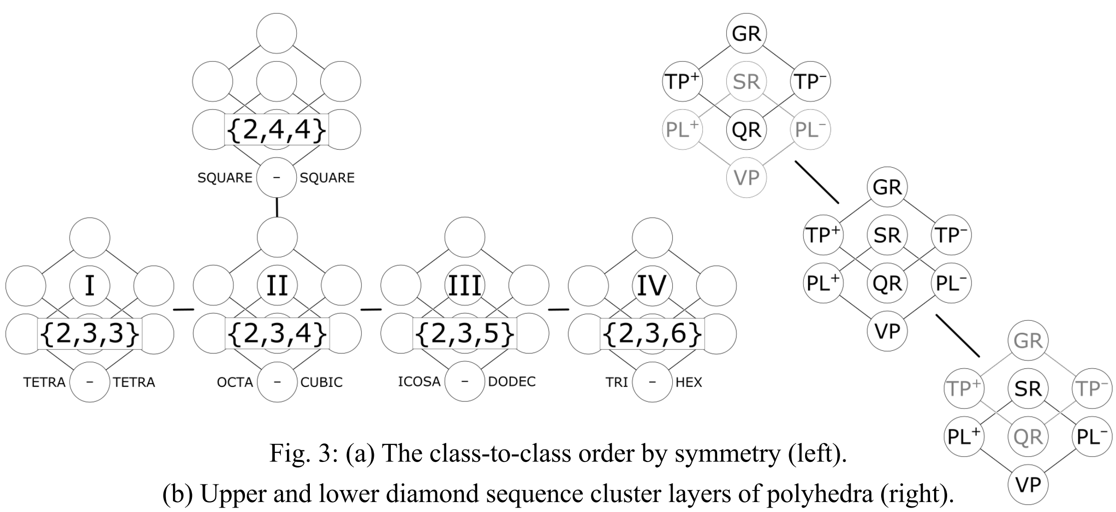


Fig. 3: (a) The class-to-class order by symmetry (left).
(b) Upper and lower diamond sequence cluster layers of polyhedra (right).

4. The revised order of the regular & semi-regular polytopes and their 2.5D schema

Inspired in part by Grünbaum and Shephard [6], my sustained exploration of the honeycombs and tessellations [7–12] led to the inclusion of virtual 0D polyhedra and polygons that acted as source PPs, and stood in clear spatial relationship to the empirical polyhedra and polygons.

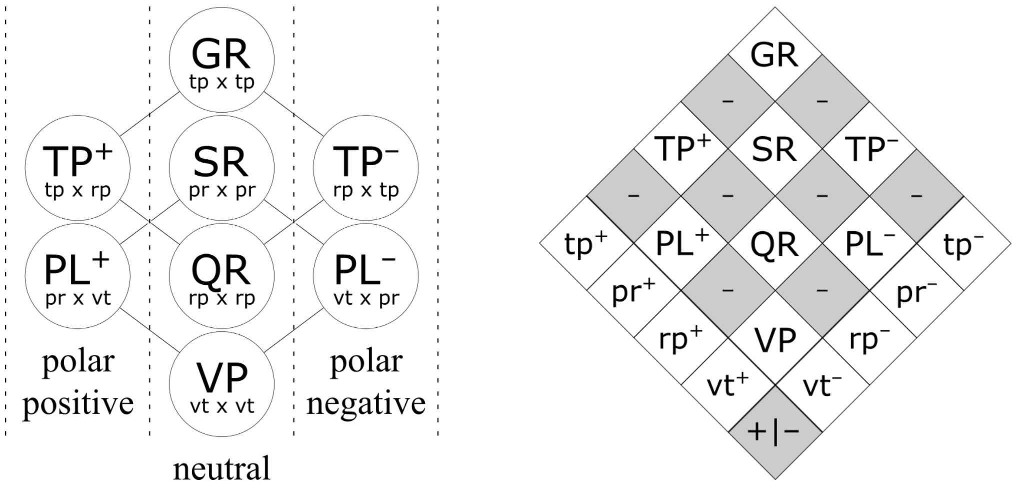
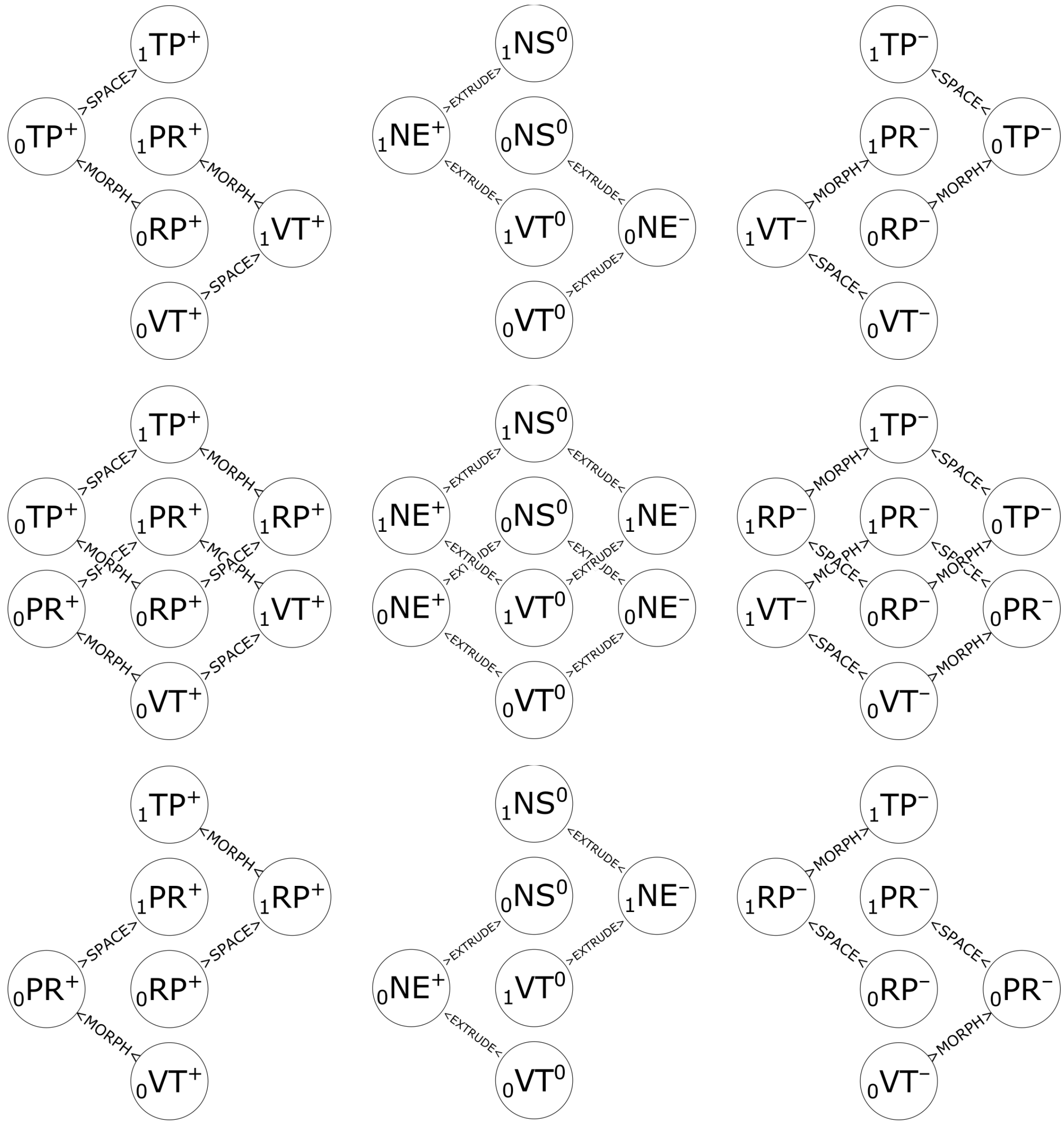


Fig. 4: (a) Qualitative differentiation of the generic order for each class into polar positive, neutral, and polar negative categories (left). (b) Matrix of positive and negative surface polytopes yielding the 8 generic PPs (right); for polytope $pp_1 \times pp_2$, balanced surface polytopes (pp_1=pp_2) produce neutral forms (center column); disjunct $pp_1 \neq pp_2$ produces polar ($PL^{+/-}$, $TP^{+/-}$) forms (left & right columns).

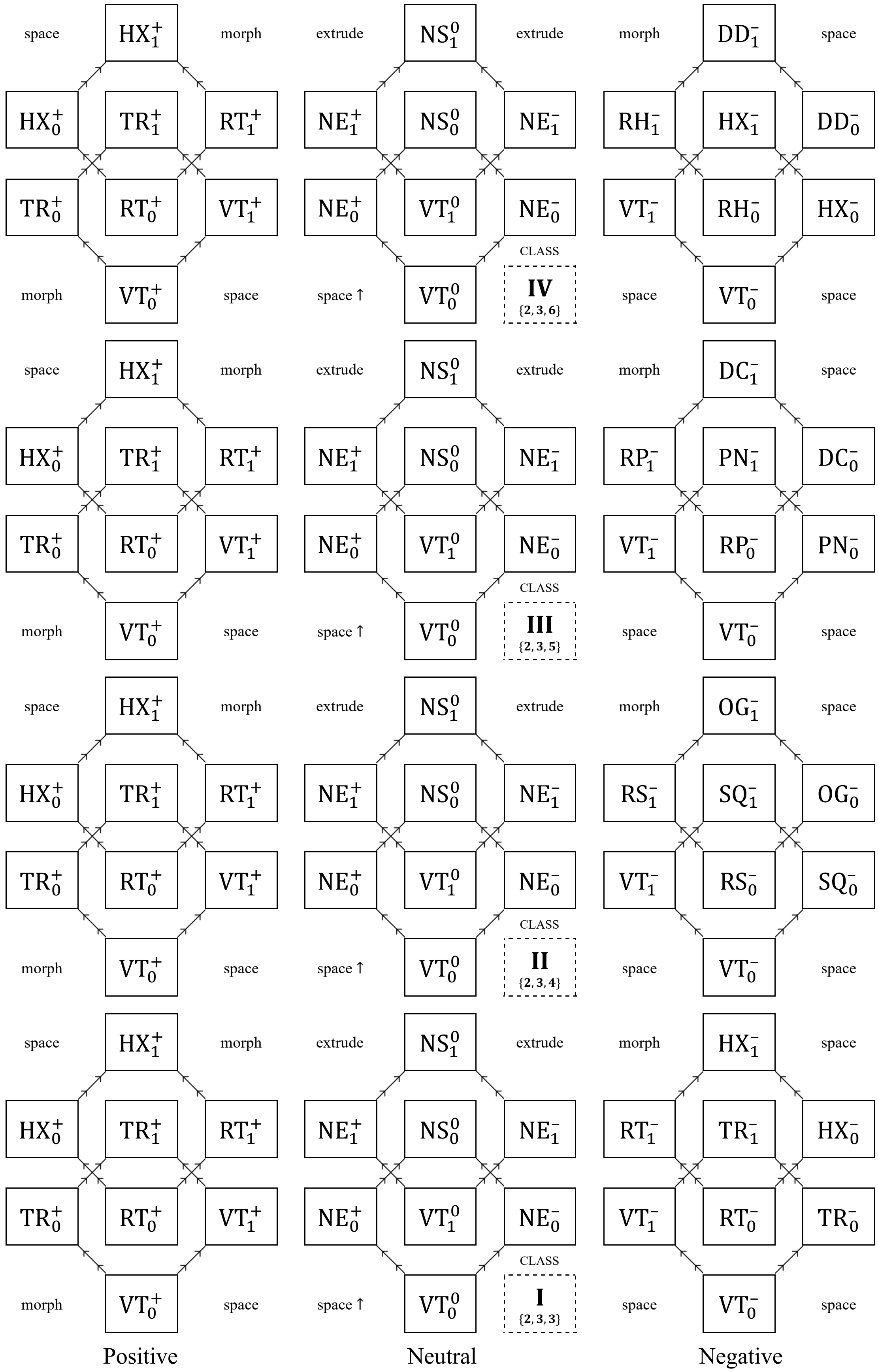
This allowed the fundamental recognition of a characteristic alternation between pairs of PPs, mediated by intermediary neutral polytopes. In this process, I apprehended an archetypal pattern of sequence clusters. These were most evident in the most numerous Class III honeycombs. This investigation led me to consider the natural relationship between the PPs of these honeycombs, as well as their neutral intermediaries; and to wonder about the relationship of the bicubic honeycomb schema of PPs to the order of single polyhedra and polygonal arrays that I had originally advanced [2]. I began to apprehend the same underlying behavior in the interrelationship of single polyhedra within each class, and in the relation of the component surface polytopes of vertices, edges, and faces to the polyhedron, or polygonal tessellation, they comprised, and the consistency of the class-to-class correspondences. I then apprehended the same sequence cluster structural motif, that the components polytopes of the singular polyhedron stood in similar relationship to that polyhedron, as the component polyhedra stood in relationship to their honeycomb. Each polyhedron/polygonal array could be recognised as the product of an interplay of polar opposites, mediated by neutral elements, just as in the honeycombs, each honeycomb consisted of an interplay of polar arrays of polyhedra, mediated by neutral polytopes. The sequence clusters evident in the honeycombs were analogued by sequence clusters of individual polyhedra in each class; and for each sequence of individual polyhedra, by sequence clusters of their surface polytopes. The virtual null honeycomb PPs that proved integral to the coherent order of honeycombs were paralleled by the virtual null VPs of the classes of polyhedra/polygonal arrays, and on the surfaces, the 0D vertices that acted as source elements of sequence clusters at the lower level of **neutral**: vertex—edge$^{+/-}$—face (2 sets of edges); and **polar**: vertex—adjoining face—separated face, or adjoining vertex—separated vertex—separated face; or at the higher level of adjoining rotated face—adjoining truncated (double frequency) face—separated truncated face, or adjoining rotated face—separated rotated face—separated truncated face. This work seeks to characterize that order, revising my earlier order of the singular polyhedra and tessellations.



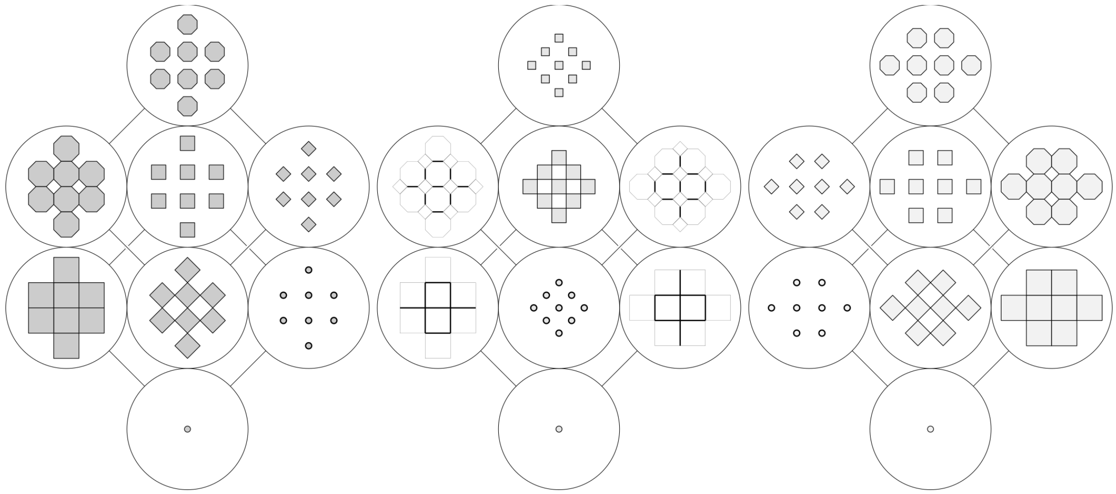
Positive Neutral Negative

Fig. 5: The generic behavior of the different classes in Fig. 6, showing the +ve, neutral, & –ve 0D, 1D, & 2D *surface polytopes* of the novel 2.5D order, characterizing each step of each sequence (middle). The regular behavior of the central neutral diagram (above/center/below) over all classes, with each sequence adjoining or separated, of ${}_{0/1}\mathrm{VT}^0$ extruding to ${}_{0/1}\mathrm{NE}^{+/-}$, then extruding to ${}_{0/1}\mathrm{NS}^0$, helps validate the order, as does the complementary nature of the +ve & –ve schema (left & right; above & below).

The main content of the paper relates to the spatial organization of the polytopes and their interrelationship according to an inherent natural harmony. Towards that end, it is mainly mediated through the visual figures and charts, rather than the text. The reader is invited to contemplate the figures at some length, and to familiarize themselves with the much earlier order of Ref. [2] (PDF 06) and the short paper Ref. [4] (PDF 67) this paper develops, which with this paper in color (PDF68), are at http://www.rmeurant.com/its/papers/polygon-1.html



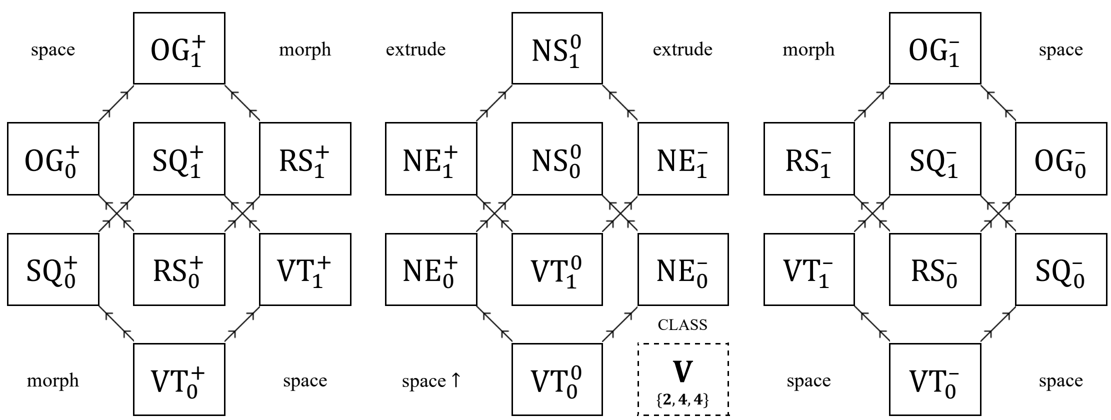
space HX_1^+ morph
HX_0^+ TR_1^+ RT_1^+
TR_0^+ RT_0^+ VT_1^+
morph VT_0^+ space
extrude NS_1^0 extrude
NE_1^+ NS_0^0 NE_1^-
NE_0^+ VT_1^0 NE_0^-
space ↑ VT_0^0
CLASS
IV
{2, 3, 6}
morph DD_1^- space
RH_1^- HX_1^- DD_0^-
VT_1^- RH_0^- HX_0^-
space VT_0^- space
space HX_1^+ morph
HX_0^+ TR_1^+ RT_1^+
TR_0^+ RT_0^+ VT_1^+
morph VT_0^+ space
extrude NS_1^0 extrude
NE_1^+ NS_0^0 NE_1^-
NE_0^+ VT_1^0 NE_0^-
space ↑ VT_0^0
CLASS
III
{2, 3, 5}
morph DC_1^- space
RP_1^- PN_1^- DC_0^-
VT_1^- RP_0^- PN_0^-
space VT_0^- space
space HX_1^+ morph
HX_0^+ TR_1^+ RT_1^+
TR_0^+ RT_0^+ VT_1^+
morph VT_0^+ space
extrude NS_1^0 extrude
NE_1^+ NS_0^0 NE_1^-
NE_0^+ VT_1^0 NE_0^-
space ↑ VT_0^0
CLASS
II
{2, 3, 4}
morph OG_1^- space
RS_1^- SQ_1^- OG_0^-
VT_1^- RS_0^- SQ_0^-
space VT_0^- space
space HX_1^+ morph
HX_0^+ TR_1^+ RT_1^+
TR_0^+ RT_0^+ VT_1^+
morph VT_0^+ space
extrude NS_1^0 extrude
NE_1^+ NS_0^0 NE_1^-
NE_0^+ VT_1^0 NE_0^-
space ↑ VT_0^0
CLASS
I
{2, 3, 3}
morph HX_1^- space
RT_1^- TR_1^- HX_0^-
VT_1^- RT_0^- TR_0^-
space VT_0^- space
Positive
Neutral
Negative



Positive Neutral Negative

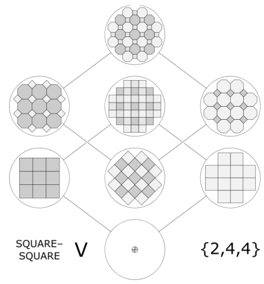
Fig. 7: Positive, neutral, and negative elements of Class V of {2,4,4} symmetry with vertex polytopes as small circles, and in the neutral elements of the PLs, faint edges of tessellation for guidance.

Fig. 6: Expansion sequences of the Positive (left), Neutral (center), and Negative (right) 0D/1D/2D vertex/edge/polygon *surface polytopes* of the primary polytopes: (a)–(d) Classes I–III of the polyhedra and Class IV tessellation (at left, facing page); and (e) Class V tessellation (below), where superscript denotes polarity, & subscript denotes spacing (0 = adjoining; 1 = adjacent with unit spacing; see Key).



Positive Neutral Negative

Nomenclature: +ve, positive; 0ve, neutral; –ve, negative; C, class; CB, cube; CO, cuboctahedron; DC, decagon (10-gon); DD, dodecagon (12-gon); E, edge; GR, great rhombic; GRCO, great rhombic cuboctahedron; GRQR, great rhombic quasiregular; HX, hexagon; NE, neutral edge; NS, neutral square; OG, octagon; OH, octahedron; PL, polar polytope; PN, pentagon; PP, primary polytope; PR, polar polygon; QR, quasiregular; RH, rotated hexagon; RP, rotated polar polygon (actually truncated); RS, rotated (truncated) square; RT, rotated (truncated) triangle; RX, rotated hexagon; Snb, snub; SnbCO, snub cuboctahedron; SnbQR, snub quasiregular; SQ, square; SS, square–square array; SR, small rhombic; SRCO, small rhombic cuboctahedron; SRQR, small rhombic quasiregular; TC, truncated cube; TH, tri–hex array; TO, truncated octahedron; TP, truncated polar polygon (double freq.); TR, triangle; VP_C, verticial polytope (Class C); VT, vertex.



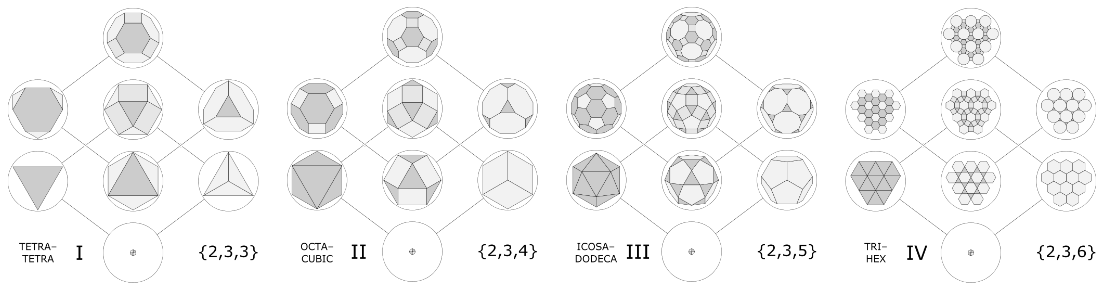
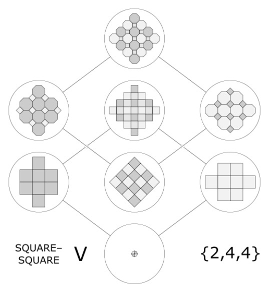


Fig. 8 (a): The five classes of regular and semi-regular polyhedra and tessellations on the Positive Axis.



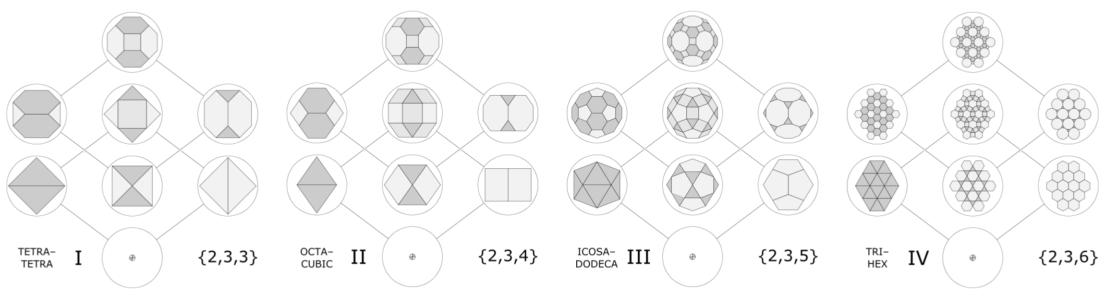
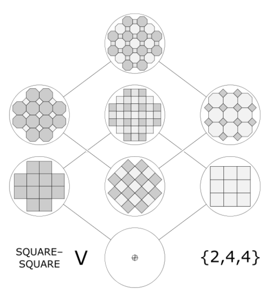


Fig. 8 (b): The five classes of regular and semi-regular polyhedra and tessellations on the Neutral Axis.



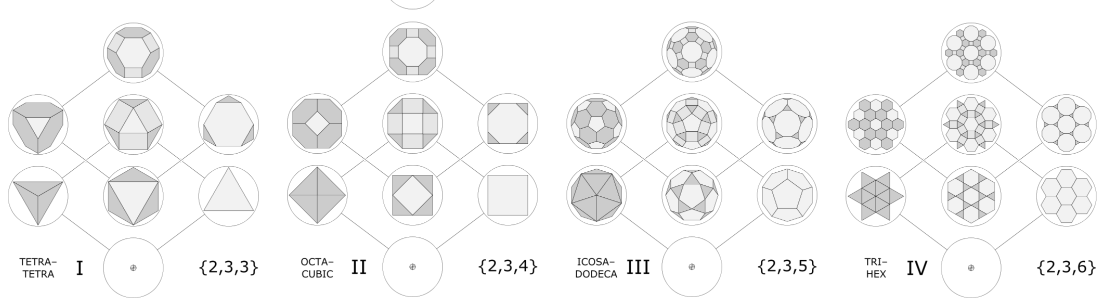


Fig. 8 (c): The five classes of regular and semi-regular polyhedra and tessellations on the Negative Axis.

NB. The behavior of unit edge separation of polytopes is easily seen in the CB—SRCO transition in Class II, where 3 SQ^{-}s meeting at a VT^{+} are separated by NSs edge-to-edge, and TR^{+} vertex-to-vertex.

The significance of this paper is that, for the first time, it situates the regular and semi-regular polyhedra and tessellations in a coherent order that enables their evolution, and that of their surface elements of vertex, edge, & polygon, to be traced under the influence of opposing poles and mediating neutral elements, and the alternating moments of polarizing and neutralizing, as they expand from primordial VP, of coincident polar & neutral VTs, to culminate in their respective GR polytope, of double frequency TPs & NSs.

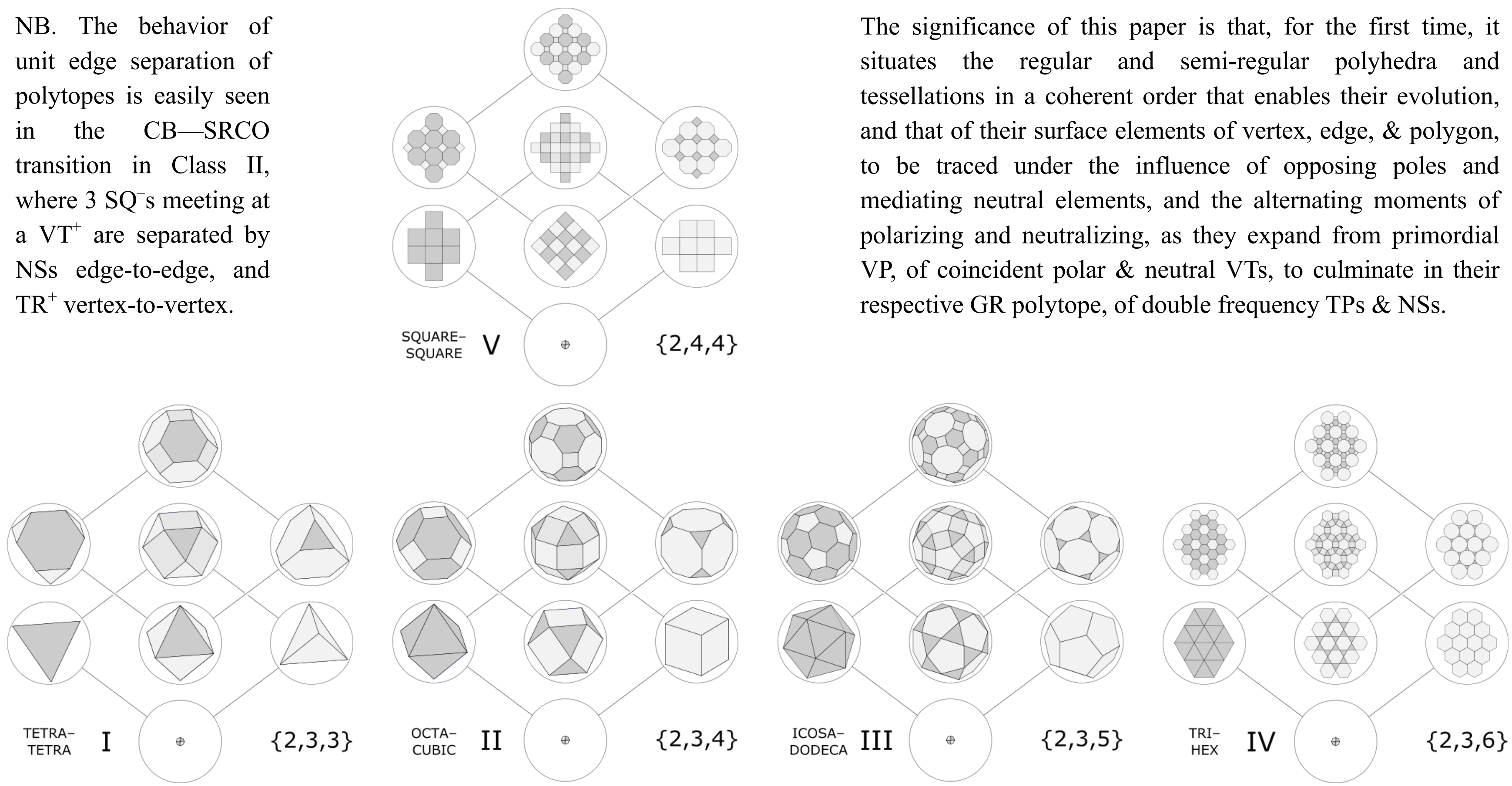
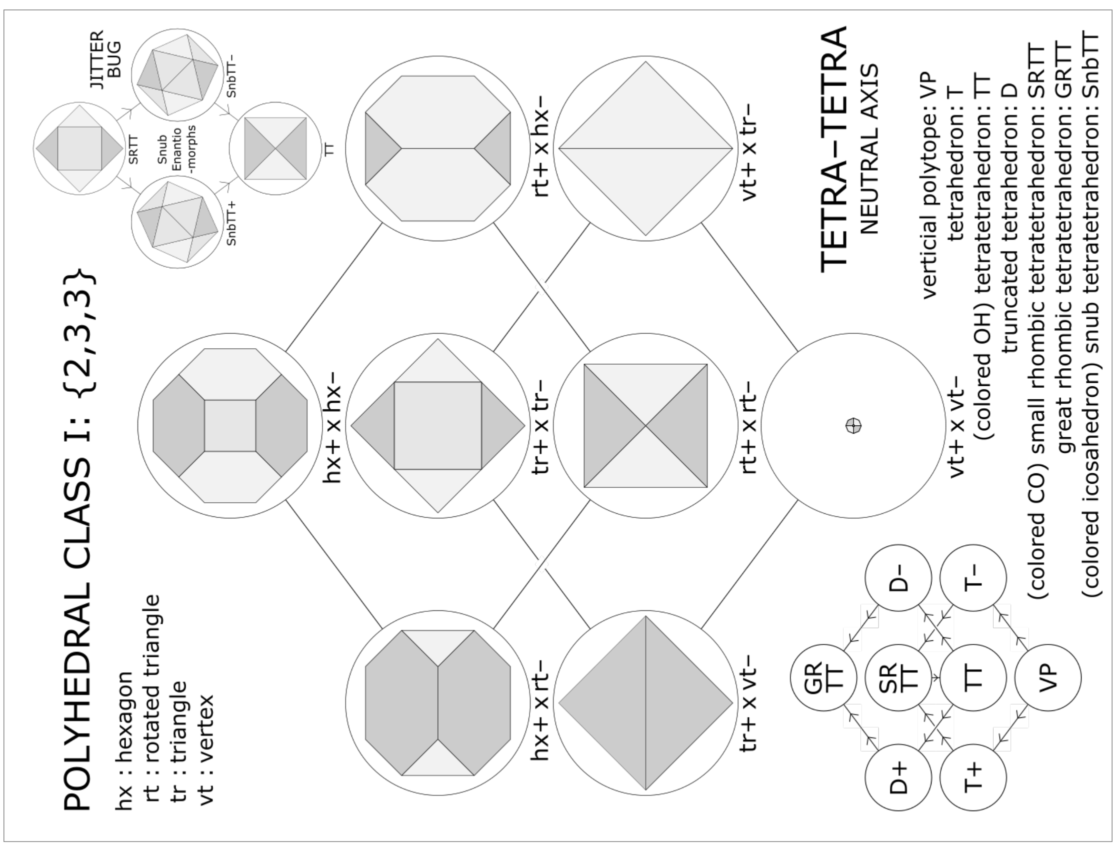
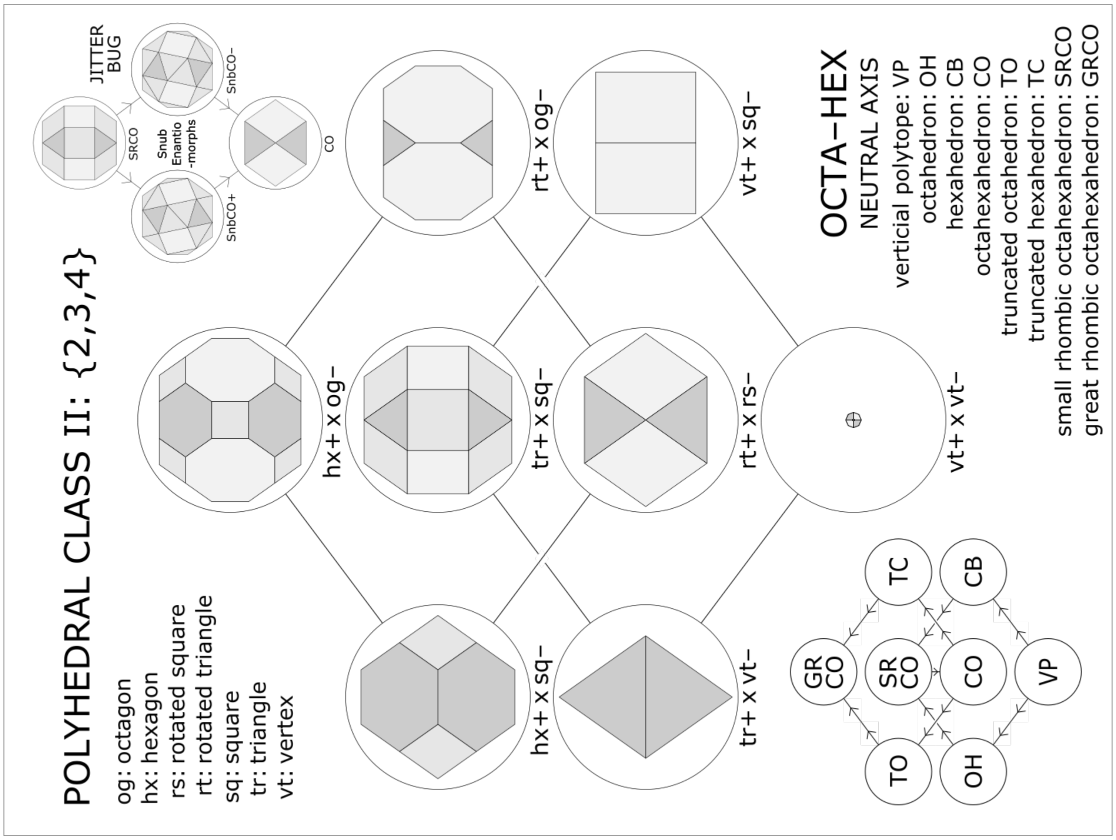
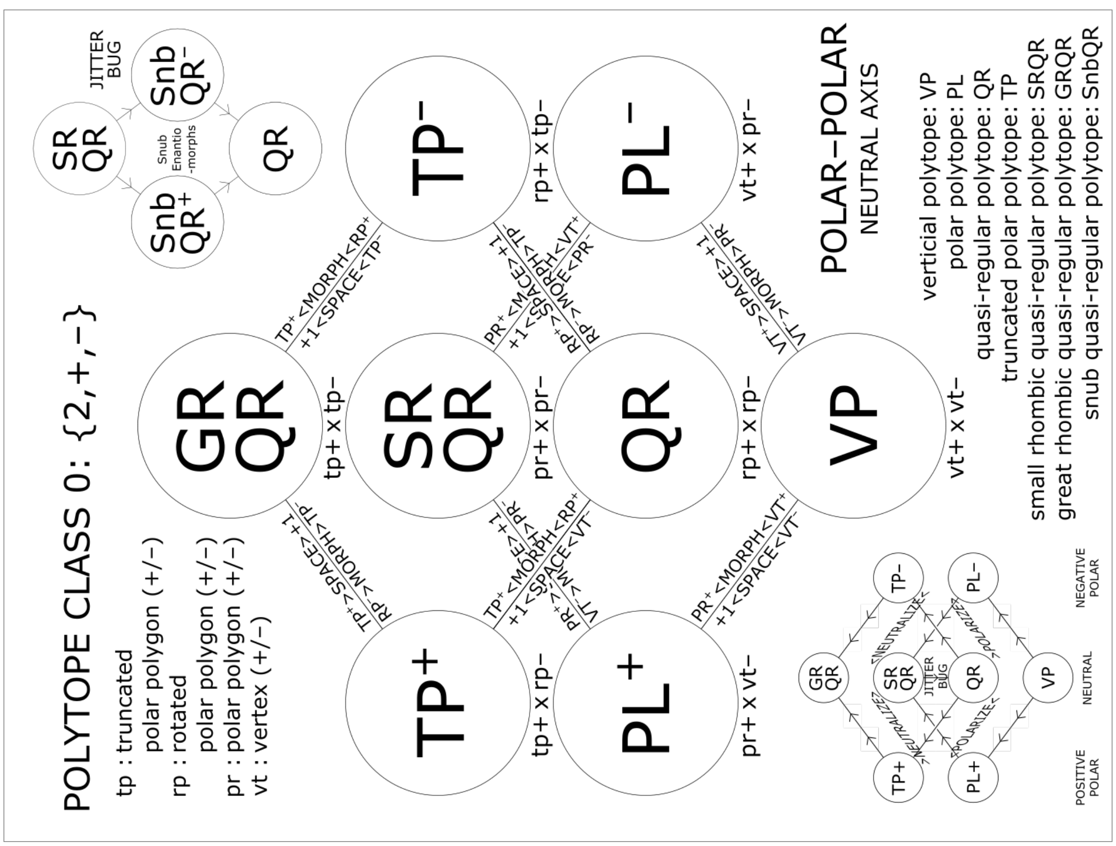


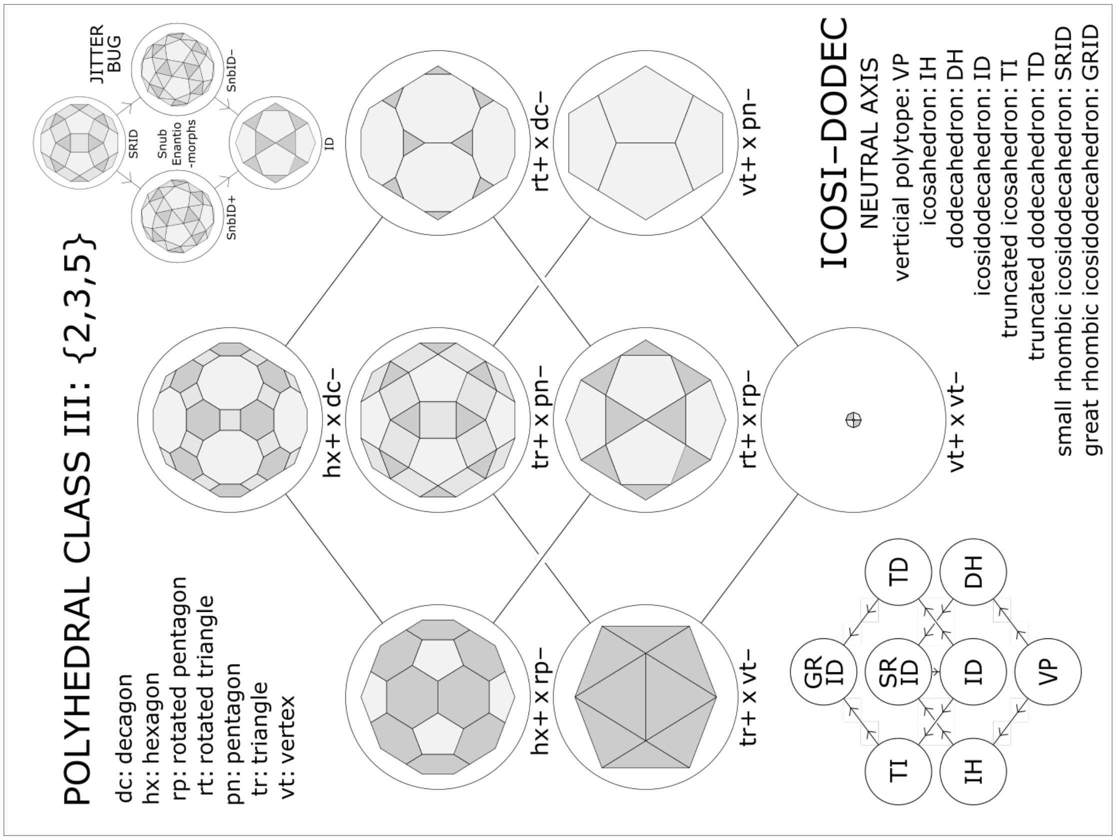
Fig. 9: The five classes of regular & semi-regular polyhedra and tessellations with polyhedra in 3D projection on offset axis, & tessellations centered on neutral axis.



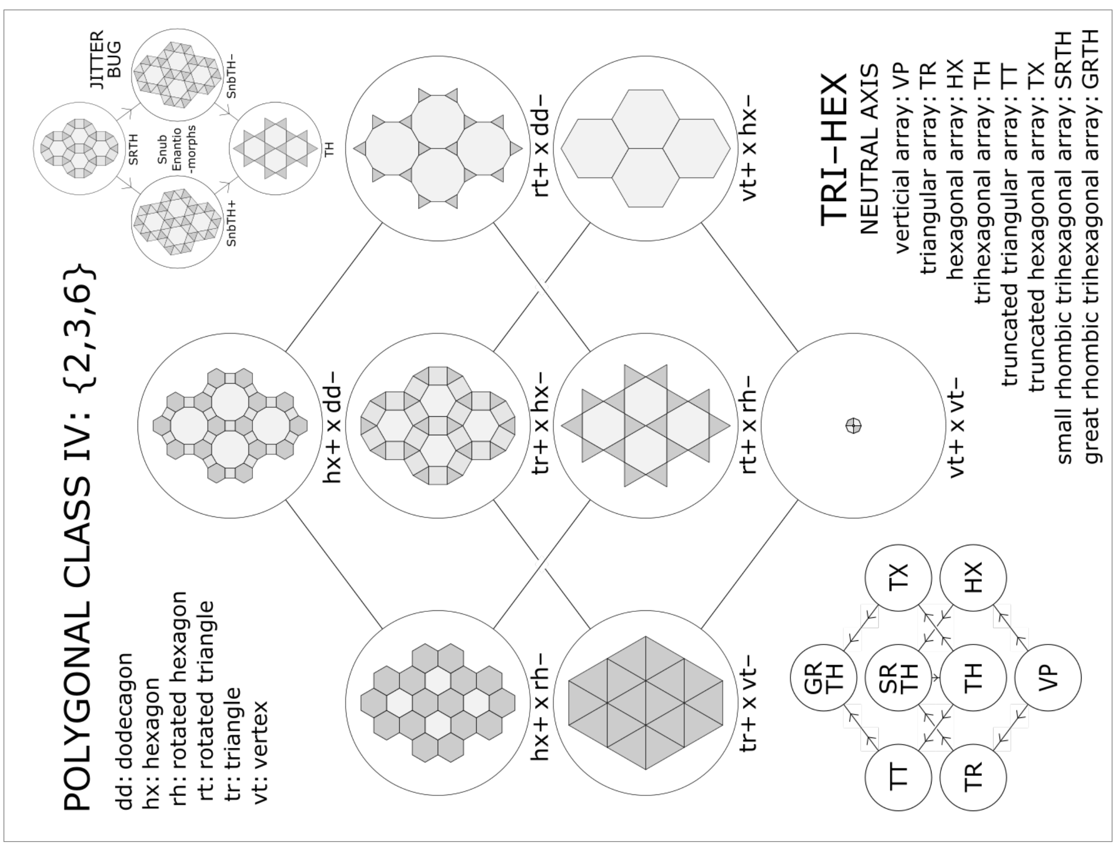


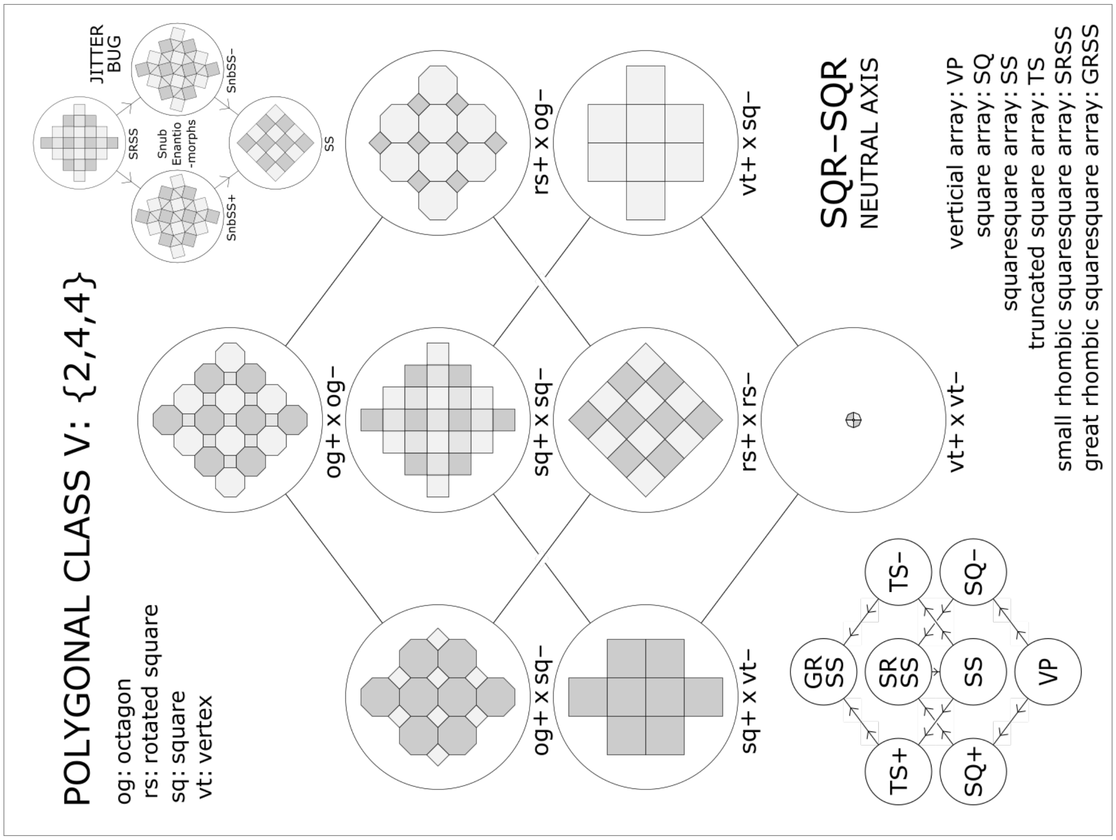
Figs. 10 (a) & (b): Information sheets of the Class I {2,3,3}and Class II {2,3,4} symmetry polyhedra.





Figs. 10 (c) & (d): Information of the generic All Class I–V and Class III {2,3,5} symmetry polyhedra.





Figs. 10 (e) & (f): Information of the Class IV {2,3,6} & V {2,4,4} symmetry polygonal tessellations.

5. Conclusion

Inspired by the recognition of the VT polytope as a PP in the honeycombs and tessellations in earlier research [7–12], the inclusion of a null VP in each class of the singular regular and semi-regular polyhedra and tessellations, and the restructuring of the generic order and each class in this paper enable the interrelationships of the elemental polytopes to be structured by class as the interplay of positive and negative poles, always mediated by neutral elements.

The 2.5D cubic schema of polyhedra in each class can be seen as cube with one long axis vertical but with specific upper & lower faces; or as a flat 2D network, and as upper & lower layer of 4 polyhedra each, or alternatively as corresponding pairs, so that the upper element or relationship always corresponds to the lower. The two layers can be seen as overlapping diamonds, whose edges correspond to the transformations of polyhedra at each step of the corresponding sequence. Each layer thus consists of a left-hand and a right-hand 2-step sequence; both start from the same neutral source polytope, transition through 2 different (complementary) polarized polytopes, and culminate in the same neutral polytope with (+&–) ve elements separating or morphing, as neutral elements extrude (Figs. 5–7). Each sequence is of 2 steps, with 2×2 sequences in each class, which can then be characterized at the polytope level, as well as the +/0/–ve level of the component surface elements of vertex, edge, & face.

In an energetic sense, the reader might imagine an epiphany of energy pouring into the universe through the point; differentiating into polar extremes that are meditated by neutrality; reunifing in a more perfect form; symmetrically and rotationally contracting through opposing transitional forms, to crystallize again in a perfectly balanced manifestation; then at that higher plane, polarizing again into complementary forms, before culminating in the highest realization possible. That cosmogony might also be apprehended in reverse, reducing from outward appearance, returning to the primordial silence within existence.

In a certain sense, the polyhedra and tessellations need not be regarded as discrete forms, but rather as constituent configurations of an orchestrated, profound harmony of space. They metamorphize or crystallize into existence as part of a coherent articulated totality richly imbued with hidden, subtle interrelationship, a non-distinct aspect of consciousness that supports the contemplation of super-empirical principles, returning the subject to the realization of higher unity within diversity, of natural harmony within the anthropogenic clutter that characterizes our fleeting existence, floating in eternity among celestial spheres.

On a more prosaic level, this order demonstrates the important structural capacity of space to be accounted for by harmonic principles that can accommodate dense texture and integrity of interrelationship. The organizing principles can be adapted and applied to structural relationships of matter and form at various scale, from nanotech, biotech, through mundane structural engineering, to Space structures, and conceivably, even at cosmological scale. This research has resulted from predominantly direct first-hand enquiry over many years into the nature of space and geometric form; the 2.5D schema thus contributes to the literature as a valuable tool. It has been a privilege to apprehend in minor degree such integrity of pattern.

Limitations of the study include the relative status of the *pr* and *rp*; Fig. 4 (b) suggests they might be swapped, which I debated. The concave polyhedra and polygons are not considered. One or two additional degenerate classes might exist, e.g., to account for the $4^2.3^3$ tiling.

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Key: Throughout, subscript = spacing (0, adjoining (sharing a vertex or edge); 1, adjacent with edge unit spacing between closest vertices or edges of polytope pairs); superscript = +/0/– polarity, *except for* neutral polar and truncated polar, where +/– denote sets of NEs that differ in orientation, according to their respective +/– polar polytope. *space* = adjoining polytopes separate ↗ or ↖ to adjacent polytopes; *morph* = polytope transforms ↗ or ↖ to next higher polytope in its sequence; *extrude* = neutral polytope expands transversely to next higher polytope in its sequence, so either neutral vertex extrudes to polar edge (1st step ↖ or ↗ of expansion sequence), or polar edge extrudes transversely to neutral square face (2nd step ↗ or ↖ of expansion sequence). *space* ↑ = neutral polytopes separate ↑ from adjoining to adjacent. Not all vertices or edges are necessarily +/0/– elements if they do not lie on the axes of symmetry, e.g., vertices of the SRCO or GRCO. Cubic 2.5D schema is generally shown in vertically compressed cuboid form. Snubs often left out.

6. References

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